On the Metric Formula for Christoffel Symbols

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1 Introduction

Christoffel symbols encode how one unit vector in a curved spacetime varies, according to the relation

$$\partial_\alpha \vec{e}_\beta = \Gamma^\mu_{\alpha\beta} \vec{e}_\mu, \tag{1}$$

where $\vec{e_\gamma}$ represents the basis vector pointing in the γ direction.

By utilizing the fact that the covariant derivative of the metric is zero and symmetries in the metric and Christoffel symbols, one can then determine the following formula of the Christoffel symbol in terms of the metric:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}). \tag{2}$$

The connection $\Gamma^\mu_{\alpha\beta}$ and the metric are such fundamental quantities, that in some ways equation (2) almost seems too complicated. Moreover, standard methods to derive the equation typically utilize tricks with index permutation and symmetries in the tensor, which does not reveal much about the underlying reason why the Christoffel symbol and the metric should be related in such a way. However, there is a picture one can have which makes more clear why one has such a combination of three derivatives of the metric (and why specifically the third term is subtracted).

2 Preliminaries

We first discuss the symmetry of the Christoffel symbol, namely

$$\partial_\alpha \vec{e}_\beta = \partial_\beta \vec{e}_\alpha. \tag{3}$$

This is true for coordinate bases. One nice way to see why this is true is to imagine embedding the manifold in a higher dimensional Euclidean space. By the Nash embedding theorem, we can do so for Riemannian manifolds. Then, denote the location of any point on the embedded manifold by $M(x^\alpha, x^\beta ...)$. We then have

$$\vec{e}_\alpha = \partial_\alpha M(x^\alpha, x^\beta ...). \tag{4}$$

Then,

$$\partial_\beta \vec{e}_\alpha = \partial_\beta \partial_\alpha M(x^\alpha, x^\beta ...), \tag{5}$$

and

$$\partial_\alpha \vec{e}_\beta = \partial_\alpha \partial_\beta M(x^\alpha, x^\beta ...). \tag{6}$$

By the equality of mixed partials, which is true for all nice manifolds we deal with, the RHS of both equations are equivalent, and so

$$\partial_\alpha \vec{e}_\beta = \partial_\beta \vec{e}_\alpha. \tag{7}$$

We now turn our attention to the metric. Given our basis vectors $\vec{e}_\alpha$, we have

$$g_{\alpha\beta} = \langle \vec{e}_\alpha, \vec{e}_\beta \rangle. \tag{8}$$

The metric is the inner product between respective basis vectors. The inner product is exactly the notion of distance that the metric prescribes. There's not any new information in equation (8). One can view it as a definition for the metric, or the inner product. I choose to view it as a definition for the inner product.

Now, the metric describes notions of measurement, while the Christoffel symbol describes components, irrespective of the manifold's notion of measurement. That is to say that we could double the "length" of the basis vectors, and the Christoffel symbol will still have the same values, since the components remain the same. The values of the metric, however, would quadruple. In order to put the notion of how a basis vector changes with respect to another direction in the same terms as what the metric describes, we are going to analyze $\langle \partial_\alpha \vec{e}_\beta, \vec{e}_\mu \rangle$. This is similar to what the Christoffel symbol encodes, as it asks the question of how much $\partial_\alpha \vec{e}_\beta$ overlaps with $\vec{e}_\mu$, but it additionally factors in the manifold's notion of measurement, which we will subsequently have to correct for.

We will show specifically why

$$\langle \partial_\alpha \vec{e}_\beta, \vec{e}_\mu \rangle = \frac{1}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}). \tag{9}$$

3 Main Visualization

Broadly speaking, when we taking the derivative of the metric, which is the inner product between basis vectors, there are two contributions coming from the derivative of each basis vector within the inner product. That is,

$$\partial_\alpha g_{\beta\mu} = \partial_\alpha \langle \vec{e}_\beta, \vec{e}_\mu \rangle = \langle \partial_\alpha \vec{e}_\beta, \vec{e}_\mu \rangle + \langle \vec{e}_\beta, \partial_\alpha \vec{e}_\mu \rangle. \tag{10}$$

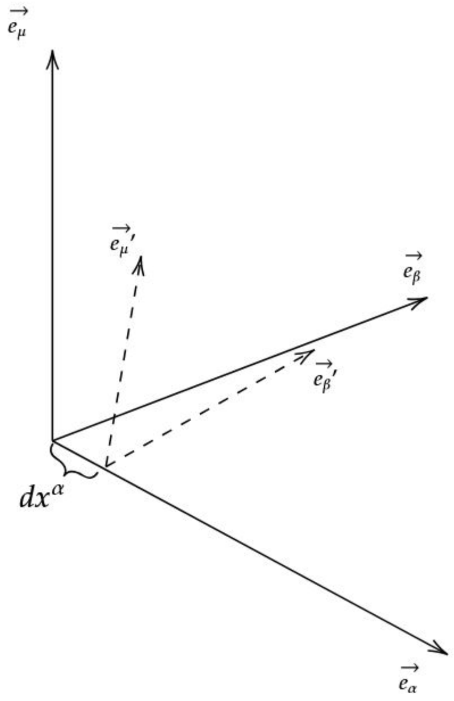

Figure 1: A sample figure of three basis vectors, and two modified vectors as one moves along the direction $\vec{e}_\alpha$.

Then, in equation (9), the first two terms both contribute a term that is equal and a term which is canceled out by the third term.

Specifically,

$$\begin{aligned}\frac{1}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}) = \frac{1}{2}(&\langle \partial_\alpha \vec{e}_\beta, \vec{e}_\mu\rangle + \langle \vec{e}_\beta, \partial_\alpha \vec{e}_\mu\rangle \\ &+ \langle \partial_\beta \vec{e}_\mu, \vec{e}_\alpha\rangle + \langle \vec{e}_\mu, \partial_\beta \vec{e}_\alpha\rangle \\ &- \langle \partial_\mu \vec{e}_\alpha, \vec{e}_\beta\rangle - \langle \vec{e}_\alpha, \partial_\mu \vec{e}_\beta\rangle).\end{aligned} \tag{11}$$

Using the symmetry established in equation (7), we have

$$\frac{1}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}) = \langle \partial_\alpha \vec{e}_\beta, \vec{e}_\mu\rangle \tag{12}$$

To visualize this, we imagine $\vec{e}_\alpha, \vec{e}_\beta$, and $\vec{e}_\mu$ all point in different directions. The following analysis still works if some point in the same direction. We visualize what happens to two of the basis vectors as we move in the direction of the third. We depict an example of this in Figure 1. As we move a distance dx^α along $\vec{e}_\alpha$, $\vec{e}_\mu$ becomes $\vec{e}_\mu\,'$, and $\vec{e}_\beta$ becomes $\vec{e}_\beta\,'$.

In our curved manifold, the basis vectors change direction as we move. Such a change, such as $\frac{\vec{e}_\mu{}' - \vec{e}_\mu}{dx^\alpha}$ in the infinitesimal limit is captured by $\partial_\alpha \vec{e}_\mu$. What we originally seek is the inner product between $\vec{e}_\mu$ and the derivative of $\vec{e}_\beta$ with respect to x^α.

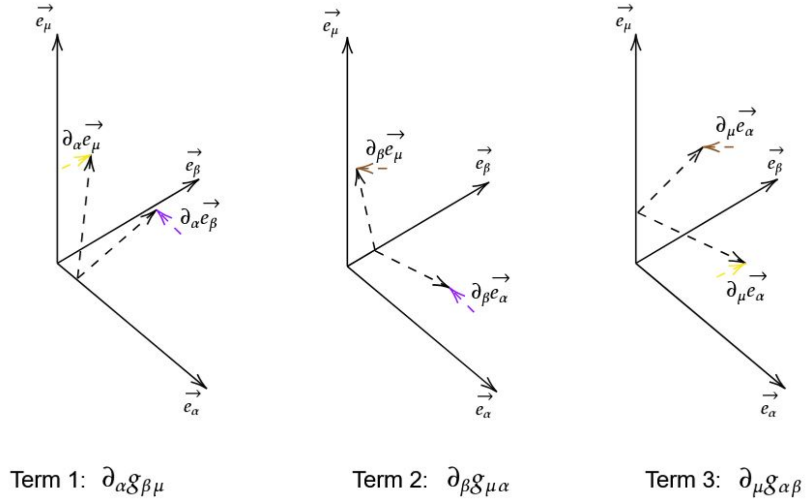

Figure 2: Depiction of the three terms in equation (12) with colors matching terms equivalent by symmetry.

For each of the three terms in equation (12), Figure 2 depicts the various derivatives in the corresponding inner product. Given the symmetry established by equation (7), we matched equivalent terms by coding them in the same color. We see that we seek just the blue arrow. The yellow arrow and the red arrow are not what we desire. It is then clear from the figure that we can achieve just the blue arrow by combining Term 1 and Term 2, both of which have a blue arrow, and subtracting Term 3, which subtracts the two undesirable arrows (taking 1/2 of the total to prevent double counting).

If $\vec{e}_\mu$ did not change, there would be no yellow arrow or red arrow, and Term 1 and 2 would be equal while Term 3 would be zero. We would then of course obtain our desired result. If $\vec{e}_\mu$ does change, either with respect to x^α or x^β, we see that the precise combination will subtract any unwanted red or yellow arrows.

4 Why the inverse metric as well?

We've established that

$$\langle \partial_\alpha \vec{e}_\beta, \vec{e}_\mu \rangle = \frac{1}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}).$$

The issue with what we currently have is that, beyond achieving the notion of overlap to one vector of the derivative of another, it additionally imbues the notion of distance in our manifold. We do not want this. We simply want to understand components, rather than getting any additional information about the length of vectors. One way of thinking about it is that by studying the derivative of the metric instead of just the basis vector, we have transformed into the space of measurement, and to get back we must transform out of it using an inverse transformation. I find it analogous to the common structure $UAU^\dagger$ often found in quantum mechanics in which one transforms, applies a specific operation, and then transforms back. In our case, the operation in question is the derivative with respect to various basis vectors. Because the metric concerns the inner product of basis vectors, which is a linear operation, the transformation in and out does not affect the derivatives which we desire.

To make things more concrete, if we have a vector $\vec{V}$, and we know its overlap with all basis vectors, assuming

$$\vec{V} = V^\gamma \vec{e}_\gamma, \tag{13}$$

then

$$\langle \vec{V}, \vec{e}_\mu \rangle = V^\gamma \langle \vec{e}_\gamma, \vec{e}_\mu \rangle = V^\gamma g_{\gamma\mu}. \tag{14}$$

Thus,

$$V^\gamma = g^{\gamma\mu} \langle \vec{V}, \vec{e}_\mu \rangle. \tag{15}$$

Thus,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}), \tag{16}$$

which is what we set out to prove. We can now dissect equation (16) into its various components. First, we seek an equation relating the Christoffel symbol and the metric (as one often has the metric only). These live in different "spaces". The Christoffel is concerned with components, while the metric is concerned the notions of distance. We transform into the distance space, bearing in mind that we'll have to transform out at the end. Then, seeking a specific overlap, we utilize the precise combination of derivatives of the metric which double count the term we seek while eliminating the other terms. Because of the double counting, we then take half this quantity. Finally, having been working in the measurement space, we transform back in the realm of components by taking the inverse transformation (provided by the inverse metric). We, then, arrive at our desired Christoffel symbol.