Deriving the Lorentz Transformation with Linear Algebra

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September 7, 2020

To begin we assume that the Lorentz transformation is a linear transformation. This prevents any unwanted accelerations between reference frames. We'll have two reference frames, S and S'. The S' reference frame is moving with respect to the S reference frame with velocity v in the x direction.

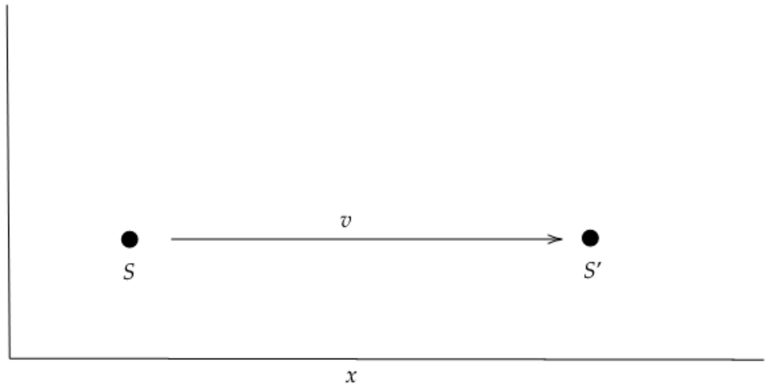

Figure [1]: The system we are working with

We let x denote the position coordinate of the S reference frame and t denote its time coordinate. Likewise x' and t' will represent the position and time coordinates of the S' frame, respectively. Thus we begin by claiming x' is some linear combination of x and t:

$$x' = ax + bt \tag{1}$$

We now question about the point whose x' coordinate is zero This is simply the origin of the S' coordinate reference frame. Since the S' reference frame is moving with a velocity v away from the S frame, we know the x coordinate of that point must be vt.

Inputting into (1) we get:

$$0 = avt + bt$$

$$-av = b \tag{2}$$

Combining (1) and (2), we may say

$$x' = a(x - vt) \tag{3}$$

Now it is a matter of expressing t' in terms of x and t. To do this, we consider the whole linear transformation which maps the vector $\begin{bmatrix} x \\ t \end{bmatrix}$ to $\begin{bmatrix} x' \\ t' \end{bmatrix}$

We can think of this as a matrix $\begin{bmatrix} a & b \\ e & f \end{bmatrix}$ such that

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{4}$$

We have already established from (2) that $b = -av$.

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} a & -av \\ e & f \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{5}$$

Our goal is the expressed e and f in terms of a and then finally derive a.

To do so we must consider the inverse transformation which maps $\begin{bmatrix} x' \\ t' \end{bmatrix}$ to $\begin{bmatrix} x \\ t \end{bmatrix}$.

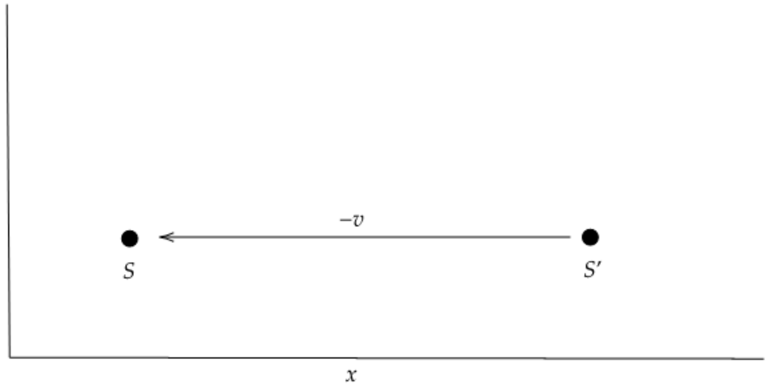

Figure [2]: The system in frame of reference of S'

Using [2], we can reason that $x = a(x' + vt')$ by stating that in the x' frame of reference, S is moving away with velocity $-v$. There is no need for our scaling factor, a, to be any different. So

$$\begin{bmatrix} a & -av \\ e & f \end{bmatrix}^{-1} = \begin{bmatrix} a & av \\ u & v \end{bmatrix} \tag{6}$$

where, again, u and v are just arbitrary constants that we shall not concern ourselves with at the moment. From this, we can rewrite the left-hand side by using the inverse matrix formula.

$$\begin{bmatrix} \frac{f}{af+eav} & \frac{av}{af+eav} \\ \frac{-e}{af+eav} & \frac{a}{af+eav} \end{bmatrix} = \begin{bmatrix} a & av \\ u & v \end{bmatrix} \tag{7}$$

Now, in order for the two top right elements of each matrix to be the same, $\frac{av}{af+eav}$ must equal av. The only way this can happen is if

$$af + eav = 1 \tag{8}$$

Allowing (8), we get

$$\begin{bmatrix} f & av \\ -e & a \end{bmatrix} = \begin{bmatrix} a & av \\ u & v \end{bmatrix} \tag{9}$$

In order for the two top left elements to be equal, f must equal a. Inputting this into (8) yields

$$e = \frac{1 - a^2}{av} \tag{10}$$

We have successfully expressed e and f in terms of a and found the transformation we were looking for, namely

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} a & -av \\ \frac{1-a^2}{av} & a \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{11}$$

To only task left is to derive the value of a. To do so we must use the second postulate of Einstein's special theory of relativity: The speed of light is unchanging in all frames of reference. Thus, if we imagine both S and S' have the same position when t and t' are 0, and a light pulse is emitted in the same direction from S as S' is moving (depicted in [3]), then we see that the position with respect to S after time t is ct, while the position with respect to S' after time t' is ct', we get

$$x = ct \tag{12}$$

$$x' = ct' \tag{13}$$

Our transformation maps any vector with position and time in one coordinate frame to another. This means that our transformation must map the vector $\begin{bmatrix} ct \\ t \end{bmatrix}$ to $\begin{bmatrix} ct' \\ t' \end{bmatrix}$. Thus

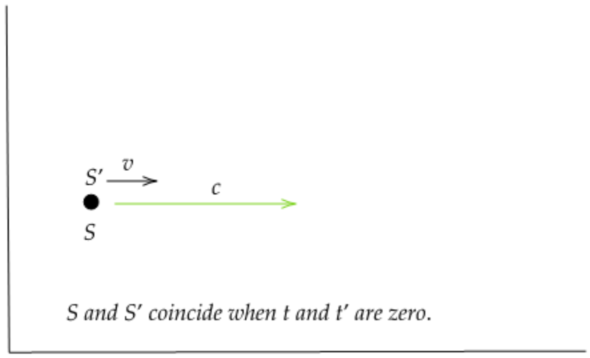

Figure [3]: A light pulse is emitted from x and $x = 0$ when t and t' are zero

$$\begin{bmatrix} ct' \\ t' \end{bmatrix} = \begin{bmatrix} a & -av \\ \frac{1-a^2}{av} & a \end{bmatrix} \begin{bmatrix} ct \\ t \end{bmatrix} \tag{14}$$

If we do the multiplication, we end up with this system of equations:

$$act - avt = ct' \tag{15}$$

$$\frac{ct - a^2ct}{av} + at = t' \tag{16}$$

If we divide (15) by c, we end up with

$$\frac{a(c-v)t}{c} = t' \tag{17}$$

Since both the left hand-sides of (16) and (17) are equal to t', we may set them equal to each other. We will skip over the algebra, but it is a good exercise to work through it. You will end up with

$$a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We have successfully derived our constant. This is often represented with the greek letter γ

We now return to our equations.

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t + \frac{x - \gamma^2 x}{\gamma^2 v})$$

We can modify the last equations by rewriting the term $\frac{x-\gamma^2 x}{\gamma^2 v}$, using the definition of γ such that

$$\frac{x-\gamma^2 x}{\gamma^2 v} = \frac{-xv}{c^2} \tag{18}$$

This simplified things greatly, and allows us to rewrite the Lorentz Transformation in the more common and well known way:

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{xv}{c^2})$$