Solution to the Brachistochrone Problem

Abstract

The Brachistochrone is a problem first posed by Johann Bernoulli in 1696. Its name derives from Ancient Greek, meaning shortest time. It imagines a system in which a bead is sliding down under gravity free of air resistance or friction along a curve in a vertical plane from position A to position B. The Brachistochrone problem is to find the shape of the curve from A to B that minimizes the time of descent. This expository article finds the expression for the total time taken on any curve from A to B and then minimizes it.

1 The Setup

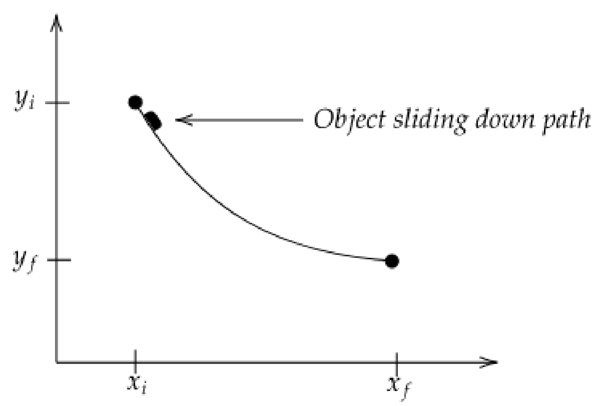


Figure [A]: A diagram of the system we are working with

Key words and phrases: Euler-Lagrange equations, Minimization, Cycloid.
Mathematics Subjects Classification (2020): 49J10.

To find the total time it takes for object to slide down the path, we can say that the total time is the integral of dt along the path.

$$T = \int_{x_i}^{x_f} dt \tag{1}$$

It is important to recognize that our bounds are in terms of x. This is because our function, y, is a function of x-not t. t is actually a dependent variable in this case. The small amount of time it takes to cross a small arc length is dependent on the small arc length which is dependent on the function. We know that

$$dt = \frac{ds}{v}.$$

This is saying that time is just distance divided by velocity. The distance is a small piece of the arc, given by

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'^2} \tag{2}$$

This is easily found by the Pythagorean theorem. To find the velocity at every point, we make use of the conservation of energy. The total energy at the starting position must be equal to the total energy at any arbitrary point on the curve. Since we are going to let go of our object at the starting point with no velocity, there will be no kinetic energy at the starting point. So, the total energy at that point is just the total gravitational potential energy:

$$u = mgh.$$

where h is the height of the object at that point, m is the mass, and g is Earth's gravitational acceleration. The height of our object is the initial y coordinate, y_i. This means the total energy at the starting point is

$$E_i = mgy_i \tag{3}$$

Now, the total energy at any arbitrary point (x, y) is the total gravitational potential energy plus the kinetic energy. We know the the gravitational potential energy is just mgy, and since the kinetic energy is $\frac{1}{2}mv^2$ where v is the velocity at that point, the total energy is

$$E = mgy + \frac{1}{2}mv^2 \tag{4}$$

By the conservation of energy E must equal E_i, so

$$mgy_i = mgy + \frac{1}{2}mv^2.$$

Solving for v, we get

$$v = \sqrt{2g(y_i - y)}. \tag{5}$$

We have now found both ds and v. Using (2) and (5), We can express dt in terms of these two quantities, and can rewrite (1) as

$$T = \int_{x_i}^{x_f} \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_i - y)}} dx \tag{6}$$

We have found the formula for the total time it takes for an object to slide down a path. Now, we must minimize this, Laconsz [1] .

2 The Minimizing

In traditional single variable calculus, you can find extrema of a function by setting the function's derivative equal to zero. This means that a small change to the input has no change to the function's output (to the first order). This is identical to saying that the ratio of the change in the function caused by the change in the input to that change in the input equals zero. This is really what the equation

$$\frac{df}{dx} = 0$$

means. We would like to do the same with the function for time that we found above, but when we try to take its derivative, we ask, "With respect to what?". Before, we were dealing with functions of a single variable, and it is easy to generalize to multiple variables. However, our time integral is a function of a path which is itself a function of x. Such functions of functions are often referred to as *functionals*. Minimizing these functionals is not as easy as before, Sussikind, Hrabovsky [5]. There is a slight caveat in this case that was not present in those previous. Let us start out generally, dealing with a functional I such that

$$I = \int_{x_i}^{x_f} F(x, y, y')dx.$$

We shall eventually apply our case of the time functional where $F(x, y, y')$ is the same as that in (6). We have a path, and we are continuously adding up a certain quantity throughout that path. If our path minimizes this integral, the deviating it slightly should have no first order effect, which we discussed above. We shall think about the discrete case with [B], to make this idea more apparent, however one should keep in their mind that there really is no difference between our discrete example and the actual continuous functions we apply our result to. So, we have some discrete path with all the points a distance Δx from each other.

Now we are going to introduce a slight deviation to our path seen in [C]. We shall shift point 3 slightly, so it goes from (x_3, y_3) to $(x_3, y_3 + dy)$

We have slightly deviated our function. This will cause a slight deviation to our functional. changing y is going to change F, so we would just like to set $\frac{\partial F}{\partial y}$

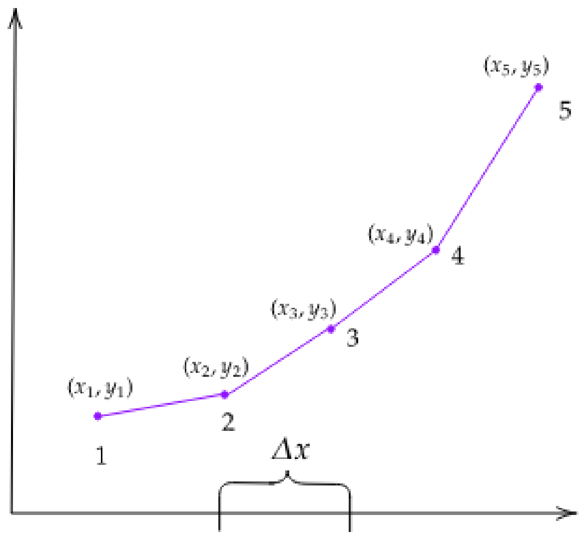


Figure [B]: A discrete path

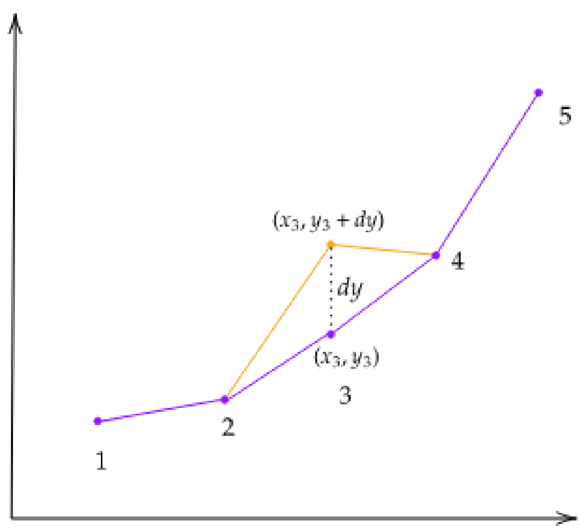


Figure [C]: The path deviated slightly

equal to 0 . There is one slight caveat that we must take into consideration. F

The reason we are setting the derivative of the integrand, F equal to zero and not just the derivative of the functional, is because Leibniz's rule allows us to move the derivative of is a function of both y and y', and changing y is also going to change y'. This means changing y affects F both directly and indirectly. We must take both of these changes into consideration.

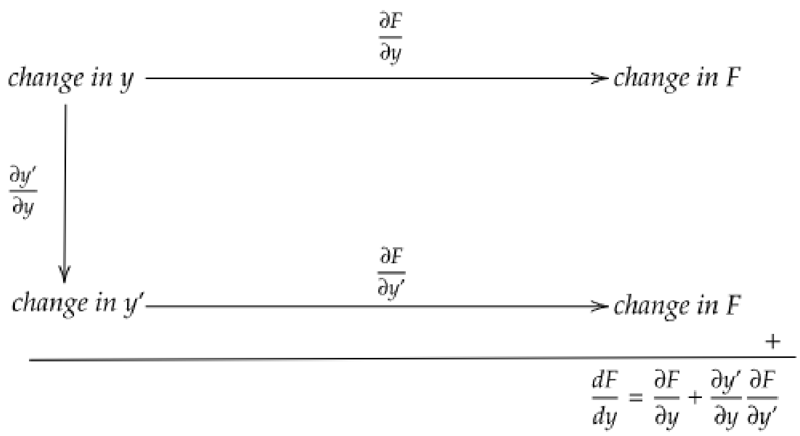


Figure [D]: Both changes in F from a change in y

From [D] we have found what both the changes in F are. We want to set this equal to zero, so,

$$\frac{\partial F}{\partial y} + \frac{\partial y'}{\partial y}\frac{\partial F}{\partial y'} = 0 \tag{7}$$

It is clear what $\frac{\partial F}{\partial y}$ means, but what about $\frac{\partial y'}{\partial y}$? It is asking about how y' changes with a slight shift in y. Clearly, there is some change, but how do we express that? We will then need to multiply that quantity by $\frac{\partial F}{\partial y'}$ which describes how F changes with a slight deviation in y'.

To do this, we will look at [B] and [C]. We see that altering y by dy makes the slope from 2 to 3 increase by a certain amount, and it makes the slope from 3 to 4 decrease by a certain amount. We will see in a moment that the slope from 2 to 3 increases by the same amount that the slope from 3 to 4 decreases. For the moment, we will take this as a given. We shall call u the amount by which the slope from 2 to 3 increased (which is also the amount the slope from 3 to 4 decreased). The net change in y' due to y is simply zero because the amounts by which the first slope increased and the second slope decreased are the same. This may lead one to believe that the quantity $\frac{\partial y'}{\partial y}\frac{\partial F}{\partial y'}$ is zero. However, the slope from 2 to 3 is at a different x value than the slope from 3 to 4. This is crucial, because if $\frac{\partial F}{\partial y'}$ is changing with x, then $\frac{\partial y'}{\partial y}\frac{\partial F}{\partial y'}$ from points 2 to 3 is not

an integral to the inside so that you are integrating the derivative of the integrand. Also, if the integral of something equals zero, then the integrand must equal zero. this allows us to say that if $\frac{\partial I}{\partial y}$ equals zero, then $\frac{\partial F}{\partial y}$ must equal zero going to be equal to $-\frac{\partial y'}{\partial y}\frac{\partial F}{\partial y'}$ from points 3 to 4. Then, the two will not cancel out, and we will have to account for that in our final solution. Because the two line segments are a distance Δx away, the amount by which $\frac{\partial F}{\partial y'}$ changed is simply $\frac{d}{dx}\frac{\partial F}{\partial y'}\Delta x$. So, if $\frac{\partial F}{\partial y'}$ is a certain value on the segment from 2 to 3, then it will have the value $\frac{\partial F}{\partial y'} + \frac{d}{dx}\frac{\partial F}{\partial y'}\Delta x$ on the segment from 3 to 4. Putting this all together, the net amount of the quantity is

$$\partial F = u\frac{\partial F}{\partial y'} - u\left[\frac{\partial F}{\partial y'} + \frac{d}{dx}\frac{\partial F}{\partial y'}\Delta x\right].$$

Note ∂F is the change in F due to the change in y' which is due to the change in y. This simplifies to

$$\partial F = -u\frac{d}{dx}\frac{\partial F}{\partial y'}\Delta x.$$

Dividing by dy, we get

$$\frac{\partial F}{\partial y} = -\frac{u}{dy}\frac{d}{dx}\frac{\partial F}{\partial y'}\Delta x = \frac{\partial y'}{\partial y}\frac{\partial F}{\partial y'}. \tag{8}$$

We have found the second quantity dictating how y changes y' and how that changes F. This is ultimately equal to the $\frac{\partial y'}{\partial y}\frac{\partial F}{\partial y'}$ quantity that we were seeking above. To finish it up, we simply figure out the value of u which is just how much the slope from 2 to 3 went up due to dy. We shall examine the slope from 2 to 3. Before the change in y, it was

$$y'([2,3]) = \frac{y_3 - y_2}{x_3 - x_2}.$$

After, it became

$$y'([2,3]) + dy'([2,3]) = \frac{y_3 + dy - y_2}{x_3 - x_2}.$$

So the difference is

$$dy'([2,3]) = \frac{dy}{x_3 - x_2} = \frac{dy}{\Delta x} = u. \tag{9}$$

Δx was substituted as the distance between the two x values. We can check to ensure that this is negative the quantity of the change in slope from 3 to 4. Before the change, it was

$$y'([3,4]) = \frac{y_4 - y_3}{x_4 - x_3}.$$

After, it became

$$y'([3,4]) + dy'([3,4]]) = \frac{y_4 - y_3 - dy}{x_4 - x_3}.$$

So the difference is

$$dy'([3,4]) = -\frac{dy}{x_4 - x_3} = -\frac{dy}{\Delta x} = -u.$$

which is indeed the result we were expecting. We now substitute the value of u found in (9) into (8) to get

$$\frac{\partial y'}{\partial y}\frac{\partial F}{\partial y'} = -\frac{d}{dx}\frac{\partial F}{\partial y'}.$$

To put it all together, we use (7) to get that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y} = 0.$$

These are the well known Euler Lagrange equations that are very useful in Lagrangian mechanics. We derived them visually and intuitively, although it may have been much easier and more rigorous to go with Lagrange's method, Roberts [3]. Either way, we have arrived at the needed result, and will we proceed to use it for our Brachistochrone.

3 Applying to the Brachistochrone

We now use the Euler Lagrange equations to solve the Brachistochrone, Philips [2]. We found in (6) the equation for F:

$$F = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_i - y)}}.$$

We now know that

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'}$$

. So, when we take the derivatives, we get

$$\frac{\sqrt{1+y'^2}}{2(y_i - y)\sqrt{2g(y_i - y)}} = \frac{1}{\sqrt{1+y'^2}\sqrt{2g(y_i - y)}}\left[\frac{y'^2}{2(y_i - y)} + \frac{y''}{1+y'^2}\right].$$

After going through the algebra, we end up with

$$y'' - \frac{1+y'^2}{2(y_i - y)} = 0.$$

The expression on the left can be rewritten as a derivative.

$$\frac{d}{dx}\left[(y_i - y)(1+y'^2)\right] = 0$$

We then integrate to get

$$(y_i - y)(1 + y'^2) = c.$$

Where c is some arbitrary constant. We can solve for y' and separate variables.

$$y' = \sqrt{\frac{c}{y_i - y} - 1}$$

$$\frac{dy}{dx} = \sqrt{\frac{c}{y_i - y} - 1}$$

$$dx = \frac{dy}{\sqrt{\frac{c}{y_i - y} - 1}}$$

$$x = \int \frac{dy}{\sqrt{\frac{c}{y_i - y} - 1}}$$

We have found an integral to express x in terms of y. All we have to do is integrate and solve. Now, this is not such a nice integral, but it is solvable.

First, we will make a simplification. We will have y_i equal to zero, so our path begins on the x axis. We are just looking for the general shape of the path, not for general functions to satisfy specific initial conditions. We then get

$$x = \int \frac{dy}{\sqrt{\frac{-c}{y} - 1}}.$$

c is just some arbitrary constant, therefore, so is $-c$. I will swap them to get rid of the negative sign. Doing some simplification, we get

$$x = \int \sqrt{\frac{y}{c - y}} dy.$$

We will now evaluate this integral. We shall start with a u substitution.

$$u = \sqrt{\frac{y}{c - y}}$$

$$du = \frac{c}{2(c - y)^2 \sqrt{\frac{y}{c-y}}} dy = \frac{(1 + u^2)^2}{2cu} dy$$

$$dy = \frac{2cu}{(1 + u^2)^2} du$$

Substituting these values in, we get

$$x = 2c \int \frac{u^2}{(1 + u^2)^2} du.$$

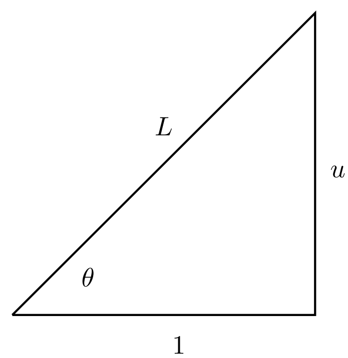
We will break this apart with partial fractions, so the integral becomes

$$x = 2c \int \left[\frac{1}{u^2+1} - \frac{1}{(u^2+1)^2} \right] du.$$

The integral of the first term is simply the $arctan(u)$. The integral of the second term is not as easy.

$$x = 2c \left[arctan(u) - \int \frac{1}{(u^2+1)^2} du \right] \quad (10)$$

To evaluate the second integral, we will make use of a helpful diagram



Here we have a right triangle with angle θ. By the Pythagorean theorem, we know

$$1 + u^2 = L^2.$$

Inside of the integral above is $\frac{1}{L^4}$. We know u is $tan(\theta)$ and L is $\frac{1}{cos(\theta)}$ Therefore,

$$du = \frac{d\theta}{cos^2(\theta)}.$$

We can use these facts to rewrite the integral above as

$$\int \frac{1}{(u^2+1)^2} du = \int cos^2(\theta) d\theta.$$

This is simply

$$\int \left(\tfrac{cos(2\theta)}{2} + \tfrac{1}{2} \right) d\theta.$$

which is

$$\tfrac{sin(2\theta)}{4} + \tfrac{\theta}{2} + c_2.$$

where c_2 is a separate arbitrary constant. I will rename the c above to c_1 to distinguish these two constants. θ is $arctan(u)$, and $sin(2\theta)$ is $2sin(\theta)cos(\theta)$. We can use the diagram above to find what $sin(\theta)$ and $cos(\theta)$ are. Rewriting the whole expression, we can say

$$\int \frac{1}{(u^2+1)^2} du = \frac{1}{2}\left[\frac{u}{u^2+1} + arctan(u) + c_2\right].$$

We can substitute this back into (10) to say that

$$x = c_1\left[arctan(u) - \frac{u}{u^2+1} + c_2\right].$$

We can replace u with its value for y and we get

$$x = c_1\left[arctan(\sqrt{\frac{y}{c_1-y}}) - \frac{(c_1-y)\sqrt{\frac{y}{c_1-y}}}{c_1} + c_2\right].$$

We have now found the equation for the quickest path. It is somewhat complicated, and x is in terms of y. Now we will make it simpler. Some of the work we did above was just to get x and y into one single equation alone, but to make this simplify things, we shall parameterize x and y. We will actually return to a previous substitution we did, but in a more direct manner. We will say

$$\sqrt{\frac{y}{c_1-y}} = tan(\alpha).$$

Therefore

$$y = c_1 sin^2(\alpha) = \frac{c_1}{2}\left(1 - cos(2\alpha)\right).$$

The variable x becomes

$$x = \frac{c_1}{2}\left(2\alpha - sin(2\alpha) + \frac{c_1}{c_2}\right).$$

We can make a final substitution and adjust the constants.

$$s = 2\alpha$$

We are now able to say that

$$x = c_1\left(s - sin(s) + c_2\right)$$

We shall let c_2 remain the same because it is still an arbitrary constant and we have not done any operations with it. c_2 only really controls the initial conditions, and we have already established that is not what is important.

Again keeping c_2 unchanged

and

$$y = c_1 \left(1 - cos(s)\right).$$

These are the equations of the cycloid, Roidt [3]. They govern how a point on a circle would move as the circle moves along a flat path. We have found the curve that minimizes the time it takes for an object to slide down that curve.

4 References

[1] C.Lanczos, The variational principles of Mechanics, 4th ed., Dover, New York, 1986

[2] J.P.Philips, Brachistochrone, Tautochrone, Cycloid – Apple of Discard, The Mathematics Teacher, 60 (5), 1967, 500-508.

[3] C.Roberts, Elementary differential Equations, CRC Press, 2018.

[4] T.Roidt, Cycloids and Paths, Portland State University, 2011.

[5] L.Susskind and G.Hrabovsky, The Theoretical Minimum, Stanford University Basic Books, 2013.