Notes on Electrostatics

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1 Introduction

This article is perhaps the most complex and involved paper I have written so far, so it is imperative that the reader follows the concepts and arguments very closely. It provides my own notes on Electrostatics, and is intended to be supplementary to the textbook *Introduction to Electrodynamics* by David J. Griffiths[1]. I will cover and address everything I found important or had a problem understanding, however, I will not review material that is already covered in the textbook. These notes are intended to be additional content to Chapter 2 of the book. As a result, many fundamental ideas will be brushed over or ignored, and more nuanced and intricate details will be at the forefront of our attention. I cannot guarantee that the topics presented will be in any particular order. The most significant result of this article is the intuitive, visual derivation of the total energy stored in a continuous charge distribution:

$$W = \int_{AllSpace} E^2 d\tau.$$

This is tackled in section 3.

2 Gauss's Law

Gauss's Law originates from the simple fact that in an electric field, the divergence *everywhere* is zero where there is no charge (or where $\rho = 0$). This fact is not by any means obvious, and is a direct result from the fact that the electric field is inversely proportional to r^2. If it were just inversely proportional to r, then that would not be true. To understand this, we must first understand the nature of divergence. Divergence is defined as the flux on an infinitesimal region divided by the volume of that infinitesimal region. If we have some infinitesimal cube with side lengths dx, dy, and dz, then the divergence is the sum of the flux out of each side all divided by $dxdydz$ or $d\tau$ for short. We will return to definition in the future. For now, we will just use this fact to show that a $\frac{1}{r^2}$ relation proves Gauss's Law.

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[1] I worked through the second edition

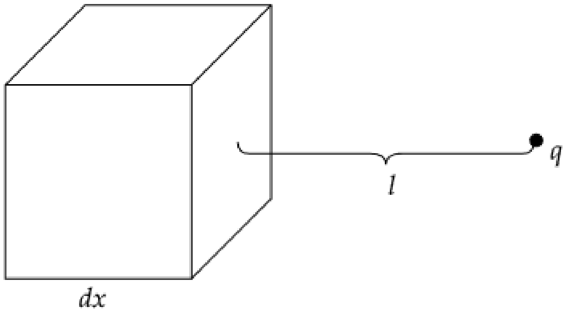

Figure 1: Divergence from a point charge outside of the infinitesimal cube

As we see in Figure 1, we have an infinitesimal region over which we will compute the flux and thus calculate the divergence. There is a point charge outside of the infinitesimal region. Since there is no charge inside the cube, ρ is zero. If we can prove this case, we can use superposition to fully prove Gauss's law. When we calculate the net flux out of each side, we get

$$F = \kappa dA \left[\frac{-1}{l^2} + \frac{1}{(l+dx)^2} + \frac{dx}{((l+dx/2)^2 + (dx/2)^2)^{\frac{3}{2}}} \right]$$

where κ is $\frac{q}{4\pi\epsilon_0}$. To find the divergence we simply divide by $d\tau$. However, one may realize that for sufficiently small values of dx, the divergence is zero. Alternatively, we could have used the same logic with a spherical $d\tau$. The sides of the volume would have no contribution to the flux because the force vector will have no orthogonal component to the area, leaving just the "front" and "back" faces. With that, it is easier to see how the $\frac{1}{R^2}$ principle is significant. Gauss's law is now proven for the special case of just one point charge, but we can use the superposition principle to generalize it to all electric fields without a charge at the particular point we are calculating the divergence. A rough explanation result comes from first examining the first two terms of our flux. We see that if it were just two terms, then out divergence would be distinctly negative, because the positive divergence originates from a force that is weaker by a $\frac{1}{dx^2}$. However, the third terms is what accounts for this. It arises from the fact there is more area affected in the flux integral the further away we are from the charge. The force weakens by r^2, but the amount of area increase by r^2, thus balancing out to a net flux (and as a result divergence) of zero.

There is also another argument we could make using field lines. Imagine, we have some charge q (Figure 2), and for every Coulomb it has, we will draw $\frac{1}{\epsilon_0}$ lines emanating from it. These lines are evenly spaced out and thus are

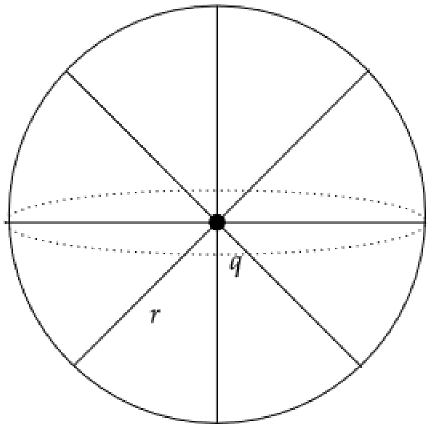

Figure 2: A charge with spherical field lines

positioned in a spherical configuration. We will define the density of these lines as the number of field lines orthogonally intersecting a unit area. Because these lines are positioned in the spherical configuration, what we could do to measure the line density is to find the total number of lines passing through the sphere divided by the surface area of the sphere. Our sphere was an arbitrary choice in the sense that we could have chosen any surface, but the sphere is special in that the line density is constant through the whole sphere. It is like measuring the mass density of a large body. Dividing the total mass by the total volume only works if the mass density is constant throughout the whole body. Likewise, dividing the total number of line intersections by the total area of the sphere only works because the line density is constant throughout the whole sphere (a result of our field line configuration). Since the sphere entraps all the lines, and the total number of lines is $\frac{q}{\epsilon_0}$, and the surface area of the sphere is $4\pi r^2$, the line density, ρ, is

$$\rho = \frac{q}{4\pi\epsilon_0 r^2}.$$

This is exactly the field strength at the surface of the sphere! We have shown that the line density at a point is equal to the field strength.

We can use this principle for flux. Roughly speaking, the line density is the number of field lines per infinitesimal area, dA. But the line density is also the field strength, so the field strength is the number of field lines per infinitesimal area, dA:

$$Field\ Strength = \frac{\#\ of\ Field\ Lines}{dA}.$$

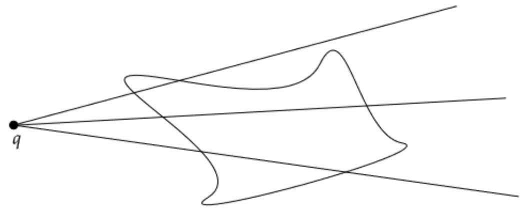

Figure 3: The number of field lines exiting a surface equal the number entering

But the flux is the Field Strength multiplied by dA [2]. Thus

$$Flux = Field\ Strength \cdot dA = \frac{\#\ of\ Field\ Lines}{dA} dA = \#\ of\ Field\ Lines.$$

We have reached a remarkable result. The flux outside of *any* surface is just the number of field lines exiting the surface (subtracting the number of lines entering). However, for any charge outside of the surface, the number of field lines entering will always equal the number exiting (Figure 3). This means the net flux is zero, further proving Gauss's Law for an area with no charge.

Now what about if there is charge at that point? Say we have a point charge q. We will draw an infinitesimal sphere around it with radius dr (Figure 4). We know the flux is just the number of field lines exiting our infinitesimal sphere, but from how we have set it up, that is $\frac{q}{\epsilon_0}$. To find the divergence, we then divide this by the infinitesimal volume of the sphere, dV:

$$\nabla \cdot \vec{E} = \frac{q}{\epsilon_0 dV}.$$

But $\frac{q}{dV}$ is just the charge density, ρ, so

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

This is Gauss's Law in its full, differential form. From here, it is just a hop skip and a jump to get to the integral form:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{ent}}{\epsilon_0}.$$

We could have also come about this by reasoning that the flux due to any one of those entrapped charges is the total number of field lines passing through the area, but because all the lines are entrapped, that number is $\frac{q}{\epsilon_0}$. Summing over all the charges, we see the net flux is the net charge divided by ϵ_0.

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[2] We do not have to worry about the dot product because the lines being orthogonal to the area is accounted for in our definition of line density

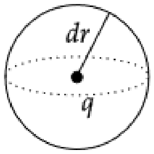

Figure 4: A point charge encompassed by an infinitesimal sphere

3 Energy for a Charge Distribution

[1] introduces this formula for the total energy of a charge distribution:

$$W = \frac{\epsilon_0}{2} \int_{all\ space} E^2 d\tau.$$

It was derived mathematically using product and integration rules. It is a very beautiful and elegant formula, which leads one to wonder if it has any intrinsic and conceptual significance. What exactly does the $E^2 d\tau$ represent? Where does it come from? This will be the topic of the section. We will derive this formula along with its region restricted counterpart. First, we will make use of the fact

$$W = \frac{1}{2} \int \rho V d\tau. \tag{1}$$

This is nicely derived in [1] using a conceptual view of potentials. The bounds of integration are not significant, so long as we integrate over all the charge, because in an area of no charge ρ is zero. What this integral essentially says, is that for every infinitesimal volume, we multiply that by the charge density and potential <u>at the center of the volume</u>. Then we sum all this together and divide by 2 to get the energy stored in the distribution.

Now, from Gauss's Law,

$$\rho = \epsilon_0 (\nabla \cdot \vec{E}),$$

so

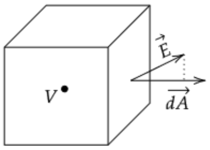

Figure 5: Flux out of $d\tau$ multiplied by V

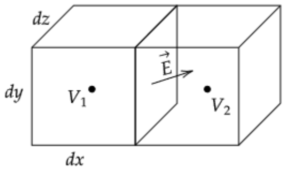

Figure 6: Positive Flux out of one side is negative flux into its neighbor

$$\rho d\tau = \epsilon_0 \oint_S \vec{E} \cdot d\vec{A} \tag{2}$$

where S is the infinitesimal surface bounding $d\tau$.

Inputting (2) into (1):

$$W = \frac{1}{2} \int V \oint_S \vec{E} \cdot d\vec{A}$$

What this essentially says is that for each $d\tau$, multiply the potential at the center by $\vec{E} \cdot d\vec{A}$ for each surface. Sum everything together and account for a factor of $\frac{\epsilon_0}{2}$. This is depicted in Figure 5.

Now, we will analyze this sum for two neighboring $d\tau$'s that share a side. It is clear that positive flux out of one of these volumes must be negative flux into the other. This is the basic principle for the Divergence theorem which allows us only to analyze the surface where there are not these cancellations. We cannot quite do that here because of the factor of V which is changing, but we can still do something similar.

In Figure 6, we see we have a side shared by two neighboring $d\tau$'s. They do not cancel because they have different values for V at their center. Their sum

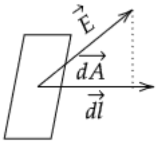

Figure 7: Summation over every adjoining square area

is then

$$S = (V_1 - V_2)(\vec{E} \cdot \vec{dA}) \tag{3}$$

where $\vec{E}$ is the value at the side, *not at any center*. This is important and is a factor in our flux integral, because it depends on E varying between sides. Now, from the definition of the potential, V, we can say

$$V_1 - V_2 = \vec{E} \cdot \vec{dl}. \tag{4}$$

Note, the reason this is not negative is because $\vec{dl}$ is pointing in the $\hat{x}$ direction which is away from the origin. Note, due to the orientation of our vectors,

$$\vec{dl} \cdot \vec{dA} = d\tau.$$

We are not invoking the second order here, because in terms of magnitude, we are multiplying together dx with $dydz$. This is an important distinction to make as we will see shortly.

Now, substituting (4) into (3):

$$S = (\vec{E} \cdot \vec{dA})(\vec{E} \cdot \vec{dl}). \tag{5}$$

We will sum this up for every square area joining two $d\tau$'s (Figure 7):

$$W = \frac{\epsilon_0}{2} \int_{Every\ Square\ Area} (\vec{E} \cdot \vec{dA})(\vec{E} \cdot \vec{dl})$$

Note that we have not yet considered sides that do not join two $d\tau$'s, namely those at the surface of our integration. For those sides, what is happening is that we are accounting for negative flux from $d\tau$'s outside the surface that do not actually exist. To undo this, we must add back that flux:

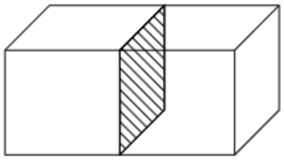

Figure 8: Two $d\tau$'s for every one square area

$$W = \frac{\epsilon_0}{2}\left[\int_{Every\ Square\ Area}(\vec{E}\cdot d\vec{A})(\vec{E}\cdot d\vec{l}) + \oint_{Surface} V(\vec{E}\cdot d\vec{A})\right]. \quad (6)$$

$\vec{E}$ in the second integral is the value at the surface, but V is the value at the center of $d\tau$ on the outside. These clarifications are important to make to be completely accurate.

In principle, (6) is correct, however it is unclear how one will go about integrating over every square area. Instead, we will reformulate the first integral, but, to achieve this reformulation, we must also alter the second integral to be accurate. What we will do is we will essentially integrate over every $d\tau$ instead of every square area. For areas not on at the surface, they will be double counted because it will have two $d\tau$'s that will account for it (Figure 8). This means we must divide our integral by two:

$$W = \frac{\epsilon_0}{2}\left[\frac{1}{2}\int_{Every\ d\tau}(\vec{E}\cdot d\vec{A})(\vec{E}\cdot d\vec{l}) + \oint_{S} V(\vec{E}\cdot d\vec{A})\right]. \quad (7)$$

Unfortunately, (7) is not quite correct. In reformulating our first integral, we have inadvertently affected our second integral which was supposed to account for areas at the surface. Our second integral (the one with $V(\vec{E}\cdot d\vec{A})$) accounted for the $d\tau$ outside the surface that the former integral counted. The new integral, however, is only counting half of the contribution of these non existent volumes. This is because our new $d\tau$ formulation is predicated on the assumption that not only do both pairs of $d\tau$'s joined by an area exist but that we are integrating over both of them. Because the $d\tau$'s beyond the surface are beyond our bound of integration, we are only accounting for half of their non existent contribution. To rectify this, we must re include the missing half of $(\vec{E}\cdot d\vec{A})(\vec{E}\cdot d\vec{l})$ at the surface:

$$W = \frac{\epsilon_0}{2}\left[\frac{1}{2}\int_{Every\ d\tau}(\vec{E}\cdot\vec{dA})(\vec{E}\cdot\vec{dl}) + \oint_S V(\vec{E}\cdot\vec{dA}) + \frac{1}{2}\oint_{Surface}(\vec{E}\cdot\vec{dA})(\vec{E}\cdot\vec{dl})\right].$$

We are finally in the home stretch. First, we combine the two surface integrals:

$$W = \frac{\epsilon_0}{2}\left[\frac{1}{2}\int_{Every\ d\tau}(\vec{E}\cdot\vec{dA})(\vec{E}\cdot\vec{dl}) + \oint_S (V + \frac{1}{2}(\vec{E}\cdot\vec{dl})(\vec{E}\cdot\vec{dA})\right]. \tag{8}$$

Now, as we clarified above, V for the surface integral represents the value of V at the center of the $d\tau$ outside the surface. However, when we add $\frac{1}{2}(\vec{E}\cdot\vec{dl})$, we reach the potential at the surface. This is due to the orientation of $\vec{dl}$ which is pointing away from the origin, towards infinity. That is just a result of how we set up the integral. Given this, we can finally rewrite (8), keeping the symbol V the same but swapping the meaning from the potential at the center of $d\tau$ outside to at the surface of our integration:

$$W = \frac{\epsilon_0}{2}\left[\int_{Every\ d\tau}\frac{1}{2}(\vec{E}\cdot\vec{dA})(\vec{E}\cdot\vec{dl}) + \oint_{Surface} V(\vec{E}\cdot\vec{dA}\right]. \tag{9}$$

There is only one last thing we must do to reach our desired integral. We will focus on the first integral once more. We will make the following change:

$$\int \frac{1}{2}(\vec{E}\cdot\vec{dA})(\vec{E}\cdot\vec{dl}) = \int \frac{d\tau}{2}(\vec{E}\cdot\hat{dl})^2. \tag{10}$$

We then remind ourselves that what this integral is meant to do is to go to each and every $d\tau$, and for every one of the six sides [3], we add $\frac{d\tau}{2}(\vec{E}\cdot\hat{dl})^2$. Thus, $\hat{dl}$ can have three possible directions (pointing in the x, y, and z directions), and of those three directions, there are two sides orthogonal to the vector which we must account for. We will begin by analyzing the case $\hat{dl}\times\hat{x}$ is zero. In Figure 9, this corresponds to analyzing sides 3 and 6. The result will be general, and a pattern will emerge.

We see in Figure 10 our two sides in more detail with their separation of dl. One thing to note is that for the side on the right, $\hat{dl}$ is equal to $\hat{x}$, but for the area on the left, $\hat{dl}$ is equal to $-\hat{x}$. This formulation is confusing, but it is how surface integrals are always represented. We have listed the value of $\vec{E}$ for each side, having the base value E as the value at the center of our $d\tau$. When we compute the sum of $\frac{d\tau}{2}(\vec{E}\cdot\hat{dl})^2$ for both sides, we get

$$\tfrac{d\tau}{2}\left[((\vec{E}+\tfrac{d\vec{E}}{dx}\tfrac{dl}{2})\cdot\hat{x})^2 + ((\vec{E}-\tfrac{d\vec{E}}{dx}\tfrac{dl}{2})\cdot -\hat{x})^2\right]$$

The negative sign in the second $\hat{x}$ will cancel in the square. Eventually, after all the cancellations, we are left with

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[3] Assuming a Cartesian system

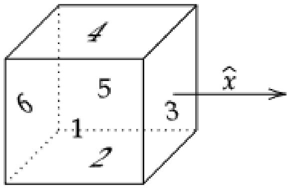

Figure 9: A cube with sides numbered (1 is the side facing the reader and 5 facing away)

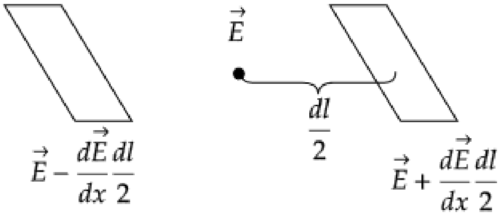

Figure 10: Two sides orthogonal to $\hat{x}$ separated by dl and their values for $\vec{E}$ labeled

$$\tfrac{d\tau}{2}\left[2E_x^2 + \tfrac{dE_x}{dx} dl^2 2\right].$$

One way to conceptually grasp why there are not cancellations but rather additions is that although $\vec{E}$ may be opposite $\vec{dl}$, so will be dV, so the product cancels negative signs, yielding a sum of squares. The second term in the brackets will go to zero because of our second order negligence. This is validates the necessity of our distinction of why variables such as dA and $d\tau$ did not invoke the second order, because right now our dl^2 invokes the second order in the direction of $\hat{x}$ which is the first time we are seeing this in the article. Letting this be zero finally yields

$$E_x^2 d\tau.$$

This was for the case of $\hat{dl}$ in the direction of $\hat{x}$, summing this for the other cases gives

$$d\tau\left[E_x^2 + E_y^2 + E_z^2\right].$$

This sums to $E^2 d\tau$! Utilizing this fact we just proved for our situation:

$$\frac{dt}{2}\left[(\vec{E}\cdot\vec{dl})^2\right] = E^2 d\tau$$

we may rewrite (9) as

$$W = \frac{\epsilon_0}{2}\left[\int_{Volume} E^2 d\tau + \oint_{Surface} V(\vec{E}\cdot\vec{dA})\right]. \tag{11}$$

This was one of the formulas [1] stated! To reach the formula of our initial motivation, we will let our surface expand towards the sphere of infinity. [1] utilizes the argument that the second integral in (11) approaches 0 because of proportionality, but we will use a different argument. The second integral was a result of having to account for what happens at the end of our surface of integration and beyond. However, if our surface of integration never ends (extends towards infinity), and there is no way to go beyond it, then we never have to worry about that. This allows us finally say that the total energy of a distribution of charge, using as surface of integration a the sphere of infinity, is

$$W = \frac{\epsilon_0}{2}\int_{All\ Space} E^2 d\tau. \tag{12}$$

This was our long sought after integral! We desired explanation of our $E^2 d\tau$ and we have found it. It came from us conducting a surface integral for an infinitesimal $d\tau$ in a volume integral. It is the result of multiplying the flux outwards of E at a point by the potential differential between that point and its neighbors. Of course, this would have been much easier to arrive at using product rules for the gradient operator, but there is a certain intuition and conceptual understanding that is only found through these more painful but well worthy efforts. From this integral, we see that the total energy must be

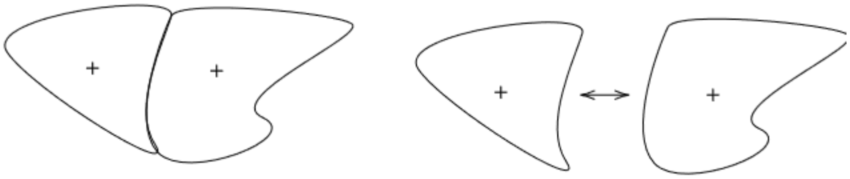

Figure 11: A portion of charge being split and repelling itself

proportional to the total charge squared. This is also clear from the definition, because doubling q both doubles ρ and V, so the energy must quadruple. We see that the total energy of a continuous charge distribution is always positive. This is a result of the fact that any continuous portion of charge of the same sign can also be split up and will thus repel each other (Figure 11). This splitting up of charge is only true for continuous distributions, because you cannot slice a point charge in half. This splitting is why the total energy of a continuous distribution is always positive.

4 Further Notes on Chapter 2

We will quickly discuss some small but significant details about Chapter 2 and its problems.

4.1 Singularities for a Continuous Distribution

In Chapter 2, there are many instances where the topic of an infinitely large sheet of surface charge density, σ, is brought up. The equation for the electric field at any height above the sheet is derived to be

$$E = \frac{\sigma}{2\epsilon_0}.$$

One may wonder why the field does not approach ∞ as our height from the sheet gets smaller and smaller. After all, the distance from the point directly beneath us should approach zero. Then, the field just from that one point should approach ∞ because the field is proportional to $\frac{1}{R^2}$ (this would be $\frac{1}{0}$). The reasons has to do with the fact that this is a continuous distribution over a surface. In essence, the $\underline{\text{point}}$ beneath us does not actually contribute to the field, because it has no area and thus no charge. The only way for something to have charge is for it to have area, courtesy of our σ formulation. The point is zero dimensional, but even a one dimensional line would have no charge, because it too has no area. This is why the field as we get closer and closer to the sheet does not reach a singularity, because that point that we get closer to does not have any charge, so we have a $\frac{0}{0}$ situation.

4.2 Gradient and Other Operators in Spherical Coordinates

4.2.1 Gradient

One key distinction between the derivative of a function with respect to some variable and the function's gradient is that the gradient gauges how some small walk in the input space affects the function, while the derivative gauges how some small change to the variable affects the function. This distinction is important, though it does not present itself plainly to Cartesian coordinates in which the variables also represent a physical length. In the case of spherical coordinates, however, the variables θ and φ represent angles, not lengths[4]. The reasoning arises from the purpose of the gradient operator. Above all, the gradient is used to compute the directional derivative (how a function changes through a certain walk in the input space in the direction of some vector). This is achieved through the dot product of the gradient of the function with the unit vector. What is necessitated out of the gradient of the function is that it describes how walks in the input space (not changes to the variables) affect the function, because we will be gauging the components of our unit vector to see the sum of how each component affects the function. In the case of spherical coordinates, the variable θ merely represents an angle. Correspondingly, $\partial\theta$ only represents a change in the angle. If we want a small walk in the input space from that, we will have to alter to it be $r\partial\theta$. A similar idea is true for φ. To calculate the gradient, we, roughly speaking, take the quotient of the small change to the function to the small walk in the input. For θ, this is

$$\tfrac{\partial f}{r\partial\theta},$$

but because r is a constant, this can be changed to

$$\tfrac{1}{r}\tfrac{\partial f}{\partial\theta}.$$

Summing this for all three components, we get

$$\nabla f = \frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial f}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}. \tag{13}$$

This result is achieved more symbolically in the first appendix of [1].

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[4] One thing to keep in mind is although those variables represent angles, their unit vectors, $\hat{\theta}$ and $\hat{\varphi}$, are ordinary orthogonal vectors

4.2.2 Divergence

Above all, what divergence aims to represent is how much a vector field is expanding outwards away from a specific point. It is often represented as $\nabla \cdot f$. This notion works perfectly fine in Cartesian coordinates, but one must be careful when dealing with spherical coordinates. One is often introduced to the dot product by being told you simply sum the product of components, but that is not quite the case for spherical coordinates (or most curvilinear coordinates in fact). If we do such, we would arrive at

$$\nabla \cdot f = \frac{\partial f_r}{\partial r} + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial f_\varphi}{\partial \varphi}^{5}.$$

Unfortunately, this is not correct, because spherical coordinates do not have the same dot product as Cartesian coordinates. There are extra terms one must add to account for the fact that the spherical unit vectors are changing with position, a detail not present with Cartesian coordinates. One way to achieve the correct result is to first convert the vectors into their Cartesian form and then compute the dot product. To do so, one must utilize the following facts:

$$\hat{r} = \begin{bmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{bmatrix}.$$

$$\hat{\theta} = \begin{bmatrix} \cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ -\sin(\theta) \end{bmatrix}.$$

$$\hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix}.$$

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \varphi}.$$

$$\vec{f} = f_r\hat{r} + f_\theta\hat{\theta} + f_\varphi\hat{\varphi}.$$

We can therefore say

$$\nabla = \begin{bmatrix} \sin(\theta)\cos(\varphi)\frac{\partial}{\partial r} + \frac{\cos(\theta)\cos(\varphi)}{r}\frac{\partial}{\partial \theta} - \frac{\sin(\varphi)}{r\sin(\theta)}\frac{\partial}{\partial \varphi} \\ \sin(\theta)\sin(\varphi)\frac{\partial}{\partial r} + \frac{\cos(\theta)\sin(\varphi)}{r}\frac{\partial}{\partial \theta} + \frac{\cos(\varphi)}{r\sin(\theta)}\frac{\partial}{\partial \varphi} \\ \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta} \end{bmatrix}.$$

$$f = \begin{bmatrix} f_r\sin(\theta)\cos(\varphi) + f_\theta\cos(\theta)\cos(\varphi) - f_\varphi\sin(\varphi) \\ f_r\sin(\theta)\sin(\varphi) + f_\theta\cos(\theta)\sin(\varphi) + f_\varphi\cos(\varphi) \\ f_r\cos(\theta) - f_\theta\sin(\theta) \end{bmatrix}.$$

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[5] Assuming f is now some vector valued function

We then compute the dot product. After distributing all the terms, we arrive at

$$\begin{gathered}
\nabla\cdot\vec{f} = \sin^2(\theta)\cos^2(\varphi)\frac{\partial f_r}{\partial r} + \sin(\theta)\cos(\theta)\cos^2(\varphi)\frac{\partial f_\theta}{\partial r} - \sin(\theta)\sin(\varphi)\cos(\varphi)\frac{\partial f_\varphi}{\partial r} + \\
\frac{\cos(\theta)\cos^2(\varphi)}{r}\frac{\partial}{\partial\theta}f_r\sin(\theta) + \frac{\cos(\theta)\cos^2(\varphi)}{r}\frac{\partial}{\partial\theta}f_\theta\cos(\theta) - \frac{\cos(\theta)\cos(\varphi)\sin(\varphi)}{r}\frac{\partial}{\partial\theta}f_\varphi - \\
\frac{\sin(\varphi)}{r}\frac{\partial}{\partial\varphi}f_r\cos(\varphi) - \frac{\sin(\varphi)\cos(\theta)}{r\sin(\theta)}\frac{\partial}{\partial\varphi}f_\theta\cos(\varphi) + \frac{\sin(\varphi)}{r\sin(\theta)}\frac{\partial}{\partial\varphi}f_\varphi\sin(\varphi) + \\
\sin^2(\theta)\sin^2(\varphi)\frac{\partial f_r}{\partial r} + \sin(\theta)\cos(\theta)\sin^2(\varphi)\frac{\partial f_\theta}{\partial r} + \sin(\theta)\sin(\varphi)\cos(\varphi)\frac{\partial f_\varphi}{\partial r} + \\
\frac{\cos(\theta)\sin^2(\varphi)}{r}\frac{\partial}{\partial\theta}v_r\sin(\theta) + \frac{\cos(\theta)\sin^2(\varphi)}{r}\frac{\partial}{\partial\theta}f_\theta\cos(\theta) + \frac{\cos(\theta)\cos(\varphi)\sin(\varphi)}{r}\frac{\partial f_\varphi}{\partial\theta} + \\
\frac{\cos(\varphi)}{r}\frac{\partial}{\partial\varphi}f_r\sin(\varphi) + \frac{\cos(\varphi)\cos(\theta)}{r\sin(\theta)}\frac{\partial}{\partial\varphi}f_\theta\sin(\varphi) + \frac{\cos(\varphi)}{r\sin(\theta)}\frac{\partial}{\partial\varphi}f_\varphi\cos(\varphi) + \\
\cos^2(\theta)\frac{\partial f_r}{\partial r} - \sin(\theta)\cos(\theta)\frac{\partial f_\theta}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}f_r\cos(\theta) + \frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}f_\theta\sin(\theta). \quad (14)
\end{gathered}$$

(14) is a very daunting equation, but luckily most of the terms cancel or reduce. After the needed simplifications, we are left with

$$\nabla\cdot\vec{f} = \frac{\partial f_r}{\partial r} + \frac{2f_r}{r} + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{f_\theta\cos(\theta)}{r\sin(\theta)} + \frac{1}{r\sin(\theta)}\frac{\partial f_\varphi}{\partial\varphi}.$$

This can be rewritten as

$$\nabla\cdot\vec{f} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 f_r + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)f_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f_\varphi}{\partial\varphi}.$$

This is the formula for divergence in spherical coordinates. Our approach using cartesian coordinates was accurate, though much of the intuition and understanding of what is actually going on in spherical divergence was lost and swept under the rug during the symbolic manipulation. We will now attempt to justify this equation in a more visual and intuitive manner. Before that, there are a few important notes to address to solidify some ideas in spherical coordinates. First, $\vec{f}$ is a function that assigns for every input of r, θ, and φ a vector that is some linear combination of $\hat{r}$, $\hat{\theta}$, $\hat{\varphi}$. The coefficients of these unit vectors for the linear combination are f_r, f_θ, and f_φ. These coefficients are not r, θ, and φ, but rather they are functions of these variables. It is important to keep separate the vector assigned by $\vec{f}$ and the vector to the point in question (though they are connected). Now, our gradient vector is also some linear combination of our spherical basis vectors (though it does not have any physical or visual interpretation). In addition, the basis vectors themselves are functions of r, θ, and φ, hence why the equations for $\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ had terms containing θ and ϕ. The fact that these basis vectors are changing is what gives rise to the added terms in our divergence formula.

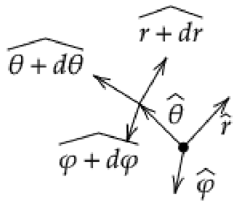

Figure 12: Spherical Coordinates rotating from walk along $\hat{\theta}$

Now, to derive our formula, we must understand the nature of divergence. Formally, it is defined as the flux per volume of some infinitesimal $d\tau$. [1] uses this definition to derive divergence in curvilinear coordinates. At its heart, what this definition aims to suggest is that as we move in the direction of one orthogonal unit vector, the function's component in that direction should also increase. This is displayed in the cartesian formula:

$$\nabla \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial y}$$

because we are gauging how each component changes as we move in the direction it is affecting. We want to generalize this notion towards spherical coordinates. The caveat is that because the unit vectors are changing, as we walk in the direction of one vector (say $\hat{\theta}$), then the component may change, but so will the unit vector it is multiplied to ($\hat{\theta}$ will become $\widehat{\theta + \partial\theta}$). We want the function to increase in the direction of $\hat{\theta}$, not $\widehat{\theta + \partial\theta}$. This alone is not enough to warrant the added terms. What is significant is that the other basis vectors, the ones that were originally orthogonal to the one under consideration ($\hat{\theta}$), are no longer orthogonal because they would have changed. In other words, another basis vector like $\hat{r}$ would change, and its component will start to affect the divergence in the direction of $\hat{\theta}$ (i.e. $\widehat{r + \partial r} \cdot \hat{\theta} \neq 0$).

Figure 12 depicts how the spherical coordinate system changes as one walks along one of the basis vectors. In this case, we walk along $\hat{\theta}$.

Figure 13 illustrates the approach we will take towards calculating the divergence. It is useful to note that only $\hat{\theta}$ and $\hat{r}$ are diagrammed, but we will still be accounting for $\hat{\varphi}$. What we want is simply to gauge how much $\vec{f}$ increases in the direction of $\hat{\theta}$ as we walk along θ. This will take the form of a ratio similar

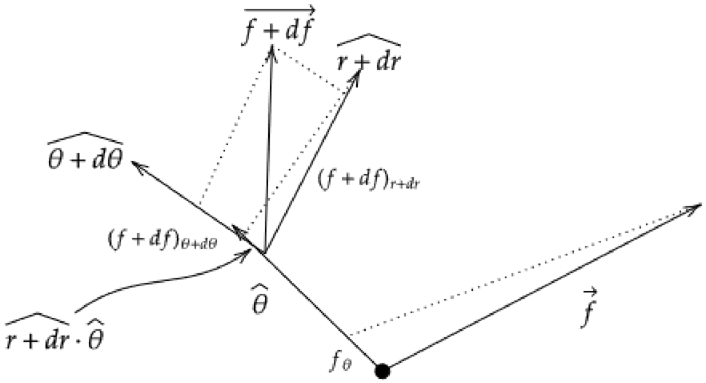

Figure 13: Changing unit vectors affecting components for divergence

to that of a derivative, with the denominator being our small step ($r\partial\theta$) and the numerator being the difference between the value before and after the step (in this case it is the dot product of $\vec{f}$ with $\hat{\theta}$). To say it more explicitly, we will let div_θ represent the θ portion of our divergence:

$$div_\theta = \frac{(\overrightarrow{f+\partial f})\cdot\hat{\theta} - f_\theta}{r\partial\theta}. \tag{15}$$

Since in this case of a small walk in the direction of $\hat{\theta}$,

$$\overrightarrow{f+\partial f} = (f_r + \frac{\partial f_r}{\partial\theta}\partial\theta)r\widehat{+\partial}r + (f_\theta + \frac{\partial f_\theta}{\partial\theta}\partial\theta)\theta\widehat{+\partial}\theta + (f_\varphi + \frac{\partial f_\varphi}{\partial\theta}\partial\theta)\varphi\widehat{+\partial}\varphi,$$

we are able to say

$$div_\theta = \frac{((f_r + \frac{\partial f_r}{\partial\theta}\partial\theta)r\widehat{+\partial}r + (f_\theta + \frac{\partial f_\theta}{\partial\theta}\partial\theta)\theta\widehat{+\partial}\theta + (f_\varphi + \frac{\partial f_\varphi}{\partial\theta}d\theta)\varphi\widehat{+\partial}\varphi)\cdot\hat{\theta} - f_\theta}{r\partial\theta}. \tag{16}$$

Now, the new basis vectors can be calculated from the old ones using their derivatives. For example, we can rewrite the first as

$$r\widehat{+\partial}r = \frac{\hat{r} + \frac{\partial\hat{r}}{\partial\theta}\partial\theta}{\sqrt{1+\partial\theta^2}}.$$

The first order negligence allows us to ignore the denominator because $d\theta^2$ is essentially zero, so

$$r\widehat{+\partial}r = \hat{r} + \frac{\partial\hat{r}}{\partial\theta}\theta.$$

From the Cartesian formula for $\hat{r}$ given above, we know

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}.$$

Utilizing the fact that the basis vectors are orthogonal (in this case $\hat{r} \cdot \hat{\theta} = 0$), we may say

$$\widehat{r + \partial r} \cdot \hat{\theta} = \partial \theta.$$

This process can be repeated for all the other combinations of derivatives of basis vectors. Normally, these derivatives are either zero or insignificant. The only other ones worth mentioning are

$$\frac{\partial \hat{r}}{\partial \varphi} = \sin(\theta)\hat{\varphi}.$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = \cos(\theta)\hat{\varphi}$$

These can also be calculated from the Cartesian formula, though one may be able to reason through it without any math utilizing the fact that $\hat{\theta}$ and $\hat{\varphi}$ are tangent to a sphere and all the basis vectors are constant in magnitude (therefore there derivatives must always be orthogonal to them).

Once one follows the same line of reasoning for the cases of θ and φ in our example of divergence for $\hat{\theta}$, they arrive at

$$\widehat{\theta + \partial \theta} \cdot \hat{\theta} = 1,$$

and

$$\widehat{\varphi + \partial \varphi} \cdot \hat{\theta} = 0.$$

Utilizing these facts, we may rewrite (16) as

$$div_\theta = \frac{(f_r + \frac{\partial f_r}{\partial \theta}\partial \theta)\partial \theta + f_\theta + \frac{\partial f_\theta}{\partial \theta}\partial \theta - f_\theta}{r \partial \theta}.$$

Simplifying and applying our first order negligence, we get

$$div_\theta = \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{f_r}{r}. \tag{17}$$

The second term can be interpreted as the result of the fact that because $\hat{r}$ is changing, it will start to affect the divergence in $\hat{\theta}$ because the new $\hat{r}$ will no longer be orthogonal to the old $\hat{\theta}$. This process and reasoning can be applied to the cases of the other two basis vectors, utilizing the two derivatives above. To get the net divergence, we simply sum up all these terms. When we do this, we arrive at

$$\nabla \cdot f = \frac{\partial f_r}{\partial r} + \frac{2 f_r}{r} + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{f_\theta}{r \sin(\theta)} + \frac{1}{r \sin(\theta)}\frac{\partial f_\varphi}{\partial \varphi}.$$

Again, this can be rewritten simply as

$$\nabla \cdot \vec{f} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 f_r + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta) f_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f_\varphi}{\partial \varphi}.$$

What we have essentially done is we have re derived the formula for divergence based off of the general notion of co-linear corresponding increase and the fact that our changing basis vectors will not stay orthogonal to their former counterparts.

References

[1] David J. Griffiths, Introduction to Electrodynamics, Prentice Hall, 1989