Pressure and Hydrostatics

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1 Introduction

Pressure is the collection of forces squeezing some object. Pressure arises from forces emanating from all directions, which is depicted in Figure 1. It is defined as force per unit area. One important thing to note is that pressure measures the orthogonal force to some area. If there is a force applied that is not orthogonal, we simply derive the pressure by viewing the orthogonal component. Furthermore, it is implied when analyzing the pressure that the force is acting on both sides of the area. If it were not, then the area section would simply displace, and there would be no pressure. Now, since there are forces from either side of the area, when choosing which components to take that are orthogonal to the area section, we must ensure the components obey the right hand rule, where the x component cross the y component equals the y component cross the z component equals the z component cross the x component.

$$P = \frac{F}{A}. \tag{1}$$

2 An Analysis of Pressure

Clearly, then, pressure is an inherently macroscopic idea. This is because fundamental particles like the electron do not have any area, and therefore, there can't be any pressure. However, you can have pressure on an exceedingly small quantity, so long as it does not have 0 area. Also, something important to note is that the forces on either side of the object in all the cases we are dealing with are going to have the same magnitude. This is because we are dealing with Hydrostatics in which the fluid does not move. That necessitates that the forces are equal because if one force were greater that its opposing counterpart, then there would be a net force and thus a net movement in the fluid. So, say we have some cube with side length, l, depicted in Figure 2. On every surface, there is a force, F, applied normal to that surface. Therefore, the pressure on the cube is

$$P = \frac{F}{l^2}.$$

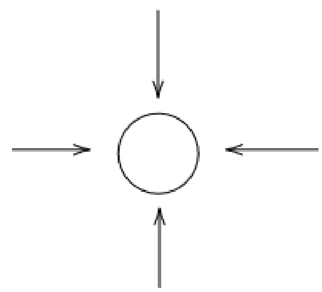

Figure 1: A depiction of pressure on a sphere

Note that we only account for the force per unit area applied to one side of the cube, and we do not add up the forces of the other five sides. This is simply how pressure is defined.

It is important to note that forces are exerted on the cube from every single possible direction. We will explain this more in the section on Pascal's principle. First, we will explain how the force is matched by every particle as we descend vertically.

3 A Brief Interlude into Particle Interactions

The particles in our fluid repel against each other with the electrostatic force governed by Coulomb's law. One important idea to make apparent is that if we have some system like Figure 4, in which a force is applied to one particle adjacent to another (we have a wall to the right just to prevent any further movements), then a force of equal magnitude will also be felt on the adjacent particle.

We see that a force, F, acts on particle A. Also, particle A and B repel each other. The force A exerts on B is h, and the force B exerts on A is g. Clearly, by Newton's third law,

$$g = h. \tag{2}$$

There are also interactions between B and the wall, but we will ignore those for the time being. Now, if the system is in static equilibrium, then A will not move. If the system weren't in static equilibrium, then A would move closer to B[1] until a point where it is at the distance such that the net force on A is 0.

[1]it could also move further away

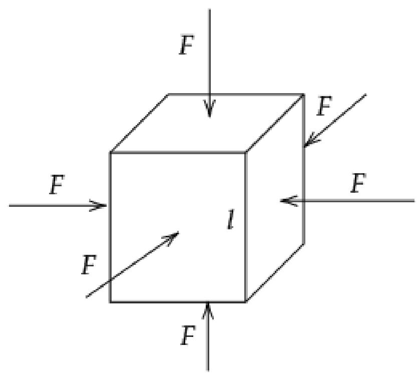

Figure 2: A cube with side length l undergoing pressure

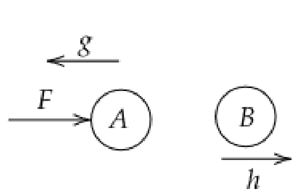

Figure 3: Two neighboring particles and a force

This implies that

$$F = g, \tag{3}$$

but because of (2),

$$F = h. \tag{4}$$

This shows that A and B both feel a force of the same magnitude in the positive (right) x direction. It is significant to note that the force A feels has a completely different origin from the force B feels. The force B feels arises from the electrostatic repulsion between the two particles, while the force A feels is just F. Because g equals F, if the wall were not in place, B would begin to move with the acceleration $\frac{F}{m_B}$, but A would stay put. This distribution of forces is crucial to our investigation, and is the key idea behind Pascal's principle, the equivalent distribution of pressure throughout a substance. Given that pressure is defined in terms of force and that area is unchanging, it almost seems clear now why that is true, but we will still probe deeper into the true nature of Pascal's principle.

4 Pascal's Principle

We have already established that if we have a substance of particles shown in Figure 5, and we apply a force, F, on the whole substance, then that force is transmitted throughout the whole medium vertically. The force is transmitted via the electromagnetic force between each particle, and nothing else[2]. Because of the floor of the container, an equal force is exerted upwards from the bottom. A similar effect happens with the walls of the container. This distribution of the force leads to a pressure on the particles. In the absence of gravity, this pressure is equal, because, as we have shown, each particle feels the same force. And if one particle feels a force in one direction, say, to the right, then it will feel an equal force in the opposite direction from the neighboring particles from that direction, so the particles to its right. This is what gives rise to the pressure on every point and explains why the pressure must be equal everywhere. It is because the force felt is equal everywhere. That is because the electrostatic reaction force is always going to match the force exerted, as seen in the previous section. This is similar, although not quite a result of, the fact that our system is in static equilibrium. Although we may not see it, the particles may initially move when the force is exerted, but they will eventually attain static equilibrium, and then, the force everywhere is equal-equal to F! Since the force everywhere is equal, and the area of a certain segment is equal, then $\frac{F}{A}$ or P is also equal everywhere, which is what Pascal's principle says. Alternatively, instead of viewing each individual particle, we can think about each layer of the fluid. Then, the force is applied to

[2]In this section we refer to particles and the pressures they feel, but this is assuming these particles have some area, which is not the case for fundamental particles like we discussed. Instead, we'll be assuming these are more macroscopic particles that do have some area

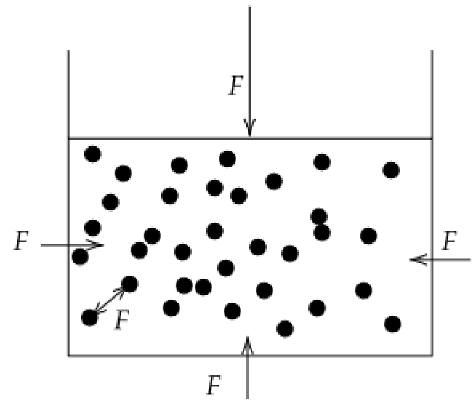

Figure 4: Our system. One property of the attainment of static equilibrium

the top layer, and regardless of the configuration and location of each individual particle, the force will be transmitted to the lower layer. If the lower layer has less area, then the force will become more concentrated and stronger on each individual particle, because multiple particles will be converging to one, and each of their forces will sum to a net force on the lower particle.

5 Further analysis of Pascal's Principle

The previous section explains why the pressure of horizontal areas at all depths is the same, but why is the pressure from every direction the same? Regardless of the orientation of your area cross section, you'll still get the same pressure result. To answer this, we'll imagine the simplest configuration for three dimensional particles we can-a cube.

As we see in Figure 5, we have a cube with the particles are the vertices. The cube itself doesn't actually exist, only the particles do. The cube just represents the length of the distances between the particles. The particles are equally spaced, so the force between them is F. We will let g denote $\kappa q_1 q_2$ in Coulomb's Law, so

$$F = \frac{g}{l^2}.$$

The area between four particles everywhere is l^2. The net force in any direction on any four particle area segment is just $4F$. Since the force and the area are the same everywhere, it is clear the pressure must be the same on every area segment. Note that it is implied there is a force matching F, perhaps from the wall or from an external force. That is what keeps this in equilibrium. Now, in reality, there are more forces acting on one particle in one direction that just

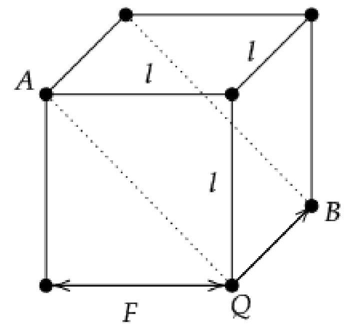

Figure 5: Equally spaced and unequally spaced particles

F between neighboring particles. There are also the diagonal forces, however those too are the same everywhere, so we are able to ignore it.

We can also analyze the pressure on the diagonal area segment with the dashed lines. The area of the segment is $l^2\sqrt{2}$. To find the force, we must find the unit vector orthogonal to the area section, and then we take the dot product of the forces with that vector. This will yield the component of the force orthogonal to the area, which is what we need to find the pressure. To find the unit vector orthogonal to the area section, we'll normalize the cross product between the vectors connecting two sides. We'll takeour base vertex to be Q. Then, the two vectors connecting it to the two other vertices on the diagonal area are

$$v\vec{_{QB}} = \begin{bmatrix} 0 \\ l \\ 0 \end{bmatrix}$$

$$v\vec{_{QA}} = \begin{bmatrix} -l \\ 0 \\ l \end{bmatrix}.$$

Then, when we take the cross product and normalize it, we get

$$\hat{v_\perp} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Now, we must take the dot product of the force vector with $\hat{v_\perp}$. Remember, the are the forces F from both sides in each direction. In other words, the net force in the x direction or y direction is zero. This doesn't mean our force vector is going to be zero. That's because of how pressure is set up-we only analyze one side of the force with the implication it is matched on the other side. When analyzing one side of the force applied to the area segment, we must ensure our components adhere to the right hand rule as we discussed above. If not, we will get faulty results because our cross product does adhere to the right hand rule. When we do this, we get that our force vector is

$$\vec{F} = \begin{bmatrix} F \\ F \\ F \end{bmatrix}.$$

Then, when we compute the dot product, we get

$$\vec{F} \ldots \hat{v_\perp} = F\sqrt{2}.$$

This is one fourth of our force component in the direction of the area. Since our procedure is the same for the other three points, the net force component in the direction of the area is $4F\sqrt{2}$. When we divide this by the area of the region, we get that the pressure is

$$P = \frac{4F}{l^2}$$

which is reassuringly exactly the same as the other areas we analyzed, further reinforcing Pascal's principle. Now, what is the side lengths aren't all the same? Would Pascal's principle still hold? Indeed it would, and we will promptly see why.

As we see in Figure 6, we have a rectangular prism configuration for our particles, with lengths of $l, h, andd$ and forces of $F_x, F_y, andF_z$, respectively. We will be analyzing the forces exerted on point Q, and then will multiply that quantity by four to obtain the net force in a particular direction on an area section.

Now, when we analyzed the cube, we only accounted for the forces pushing over the length l and not any diagonal forces. In this case, however, it will actually be more accurate to account for these diagonal forces. For example, one of these forces might be the force from the bottom left corner to Q. This would have a magnitude

$$F = \frac{g}{h^2 + l^2}. \tag{5}$$

Note that g will be the same in all particle interactions. This is just because a fluid is made of the same particles/molecules. Also, if we want to see how this diagonal force vertically pushes Q, we'll have to take the component of the force vertically, which means we'll have to multiply by the sine of the Tangle between the diagonal and the horizontal. This gives an idea of what these forces look like. What we are going to do now is find the total vertical force and horizontal force on Q and then divide those quantities by their respective areas.

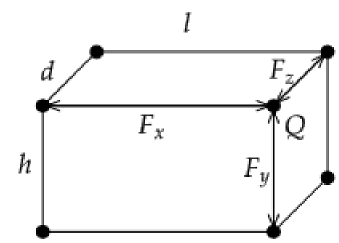

Figure 6: Unevenly spaced particles

This will show us the difference in the pressure, which we hope will be zero or insignificant. First, we find the total vertical force on Q. We already know the primary force will be $\frac{g}{h^2}$. Also, we can see from Figure 6 that there are two diagonal forces acting on Q, one with the magnitude found in (5), and another with the magnitude $\frac{g}{h^2+d^2}$. To get the vertical component, we must multiply by the sine of the angle. Finally, the is the force by the point completely opposite Q, on the back face towards the bottom left in our perspective. We must find its magnitude and multiply it by the sine of the angle. When we do all of this, we arrive at

$$\frac{F_y}{4} = \frac{g}{h^2} + \frac{gh}{(h^2+l^2)^{3/2}} + \frac{gh}{(h^2+d^2)^{3/2}} + \frac{gh}{(h^2+l^2+d^2)^{3/2}}. \tag{6}$$

F_y is divided by 4 because the vertical force on Q is only one fourth of the total vertical force on the top area section. We can then find the total horizontal force on Q. It follows nicely from symmetry that that is

$$\frac{F_x}{4} = \frac{g}{l^2} + \frac{gl}{(h^2+l^2)^{3/2}} + \frac{gl}{(l^2+d^2)^{3/2}} + \frac{gl}{(h^2+l^2+d^2)^{3/2}}. \tag{7}$$

We can then get the pressures by dividing both (6) and (7) by dl and dh, respectively. When we do that, we get

$$\frac{P_y}{4} = \frac{1}{dl}\left(\frac{g}{h^2} + \frac{gh}{(h^2+l^2)^{3/2}} + \frac{gh}{(h^2+d^2)^{3/2}} + \frac{gh}{(h^2+l^2+d^2)^{3/2}}\right) \tag{8}$$

$$\frac{P_x}{4} = \frac{1}{dh}\left(\frac{g}{l^2} + \frac{gl}{(h^2+l^2)^{3/2}} + \frac{gl}{(l^2+d^2)^{3/2}} + \frac{gl}{(h^2+l^2+d^2)^{3/2}}\right). \tag{9}$$

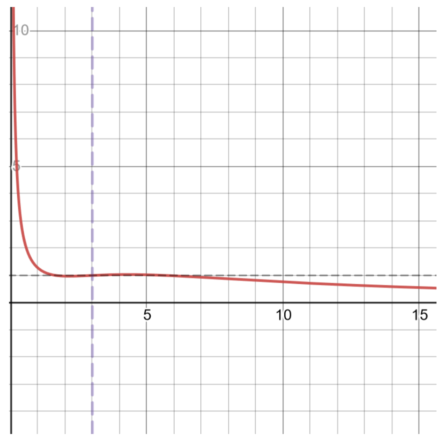

Figure 7: Our more accurate model

We can then divide (8) by (9) to obtain the ratio between the two pressures:

$$\frac{P_y}{P_x} = \frac{\frac{1}{h} + \frac{h^2}{(h^2+l^2)^{3/2}} + \frac{h^2}{(h^2+d^2)^{3/2}} + \frac{h^2}{(h^2+l^2+d^2)^{3/2}}}{\frac{1}{l} + \frac{l^2}{(h^2+l^2)^{3/2}} + \frac{l^2}{(l^2+d^2)^{3/2}} + \frac{l^2}{(h^2+l^2+d^2)^{3/2}}} \tag{10}$$

(10) looks very unsettling, however it is needed to be as accurate as possible. This added accuracy will yield a more reassuring result to the constancy of pressure as opposed to the result you would get if you had stuck to neighboring forces, which yields a ratio of $\frac{l}{h}$. We will imagine that we hold l constant, and, for simplicity, d equals l. We then vary h and see how that affects the ratio of pressures. Figure 7 depicts our current model from (10) while Figure 8 depicts what the model would've been if we had stuck with just $\frac{l}{d}$.

The lines drawn represent the line $x = l$ and $y = 1$. It is clear that is Figure 7, the graph doesn't diverge too quickly away from the line $y = 1$, which means that for a sizable neighborhood near $x = l$, the pressures are the same (more or less). Also, we could evaluate numerically the derivative of the graph when h equals l. For larger and larger values of l, the derivative more and more approaches 0. This result is much harder to see in Figure 8 and tells us our added accuracy was worth it. Also, the case from Figure 5 can be seen if we just look at when $x = l$. There, the pressure is exactly 1 which holds with what we found earlier. This further prove Pascal's principle because the pressure might as well be constant for our particle interactions. We see that for significantly different values for h and l, the pressure changes (we can ignore when $h = 0$ because that yields a singularity since we're dividing by 0) slightly, however in a fluid, the particles are so packed together, it's extremely hard for the values of h and l to change much. Also, we must note that Pascal's Principle applies to classical mechanics which makes use of many approximations of the true nature of the particle interactions, and this graph shows how the approximation is very good, but not exact. One extra thing to account for is this with more macroscopic

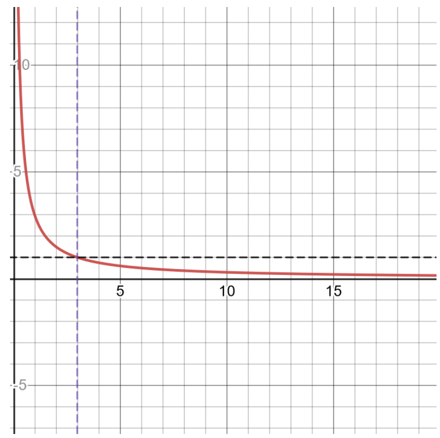

Figure 8: Our less accurate model

areas, there will be a large collection of particles. The force will be proportional to the area because more area means more particles and more particles means more forces applied orthogonal to the section. This just reinforces what we've always discussed.

Now that we have proved Pascal's Principle, one may wonder the effect of gravity is on the overall system. After all, the gravitational force should have an effect on the forces felt on different points. We will now discuss these systems.

6 Hydrostatic Pressure with Gravity

We will start by simplifying our system greatly, to the point of one dimensional stacked cubes. This allows us to see more clearly before trying to tackle larger, more sophisticated systems.

We have these cubes with side length dl and mass dm stacked. We are using differentials to give the appearance that these are sufficiently small cubes to do calculus with. We then see that there is a force applied to the top cube, F, and each point also feels its gravitational force, gdm. The top cube then exerts the force $F + gdm$ on the lower cube. Because the force A exerts on B is the force B exerts on A, the top cube will feel a force upwards on magnitude $F + gdm$. This keeps the balance, because both in the up and down direction does the top cube feel a force $F + gdm$. As we proceed down, we see that the second to top cube feels the force down from the top cube plus its gravitational force, yielding $F + 2gdm$. By the same token, it feels the same force upwards. This pattern continues. We will now analyze the pressure as a function of the height, y. For simplicity, we will have the line $y = 0$ be at the top of the top cube. From the analysis of the force, we see

$$f(y) = F - \frac{ygdm}{l}.$$

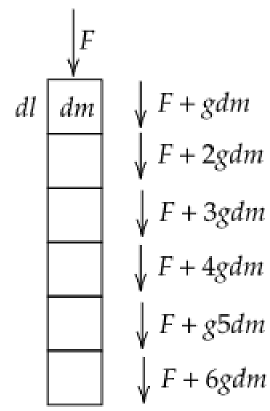

Figure 9: Infinitesimal cubes stacked atop each other

Since

$$P = \frac{F}{A} = \frac{f(y)}{da}$$

and

$$da = dl^2,$$

we can say

$$P(y) = \frac{F}{dl^2} - \frac{ygdm}{dl^3}.$$

Since dl^3 is dv, the volume, and $\frac{dm}{dv}$ is the density, ρ, we can say

$$P(y) = \frac{F}{da} - \rho g y \tag{11}$$

If we differentiate (5) with respect to y, we get

$$\frac{dP}{dy} = -\rho g \tag{12}$$

which is the equation for Hydrostatic Pressure. One important thing to note is that the previous sections did away with the notion of vertically stacked particles, while this section utilized it. The usage was simply to show the gaining of a negative vertical component as we progressed to further and further y levels and was not completely needed, although it did make many calculations very simple. This gaining of the vertical component gdm is what increased the pressure as we decreased y values. In fact, it may be useful to think of the infinitesimal layers of the fluid gradually transferring and contributing to the force. It may be useful to abandon our usual theme of particle views momentarily, because the view or layers provides the same insight while generalizing and side stepping the complicated, less significant interactions between each particle. In this sense, we can think of each cube as a layer to the liquid. Granted, the layer would look more like an extremely short rectangular prism, but the calculations would look the same as with the cube.

7 Archimedes' Law

Archimedes' Law states

Law 1 *The upward buoyant force exerted on a body by a fluid is equal to the weight of the amount of fluid displaced by the body.*

We can visualize this with Figure 10. [a] depicts a static fluid, and we have carved out some area whose weight has magnitude F. When we drop an object into the fluid such that the object displaces exactly that section as seen in [b], then the buoyant force upwards that the object experiences is exactly equal to F, the weight of the displaced fluid. If the buoyant force is greatly than or equal to the gravitational force of the object, it will float. Otherwise, it will sink.

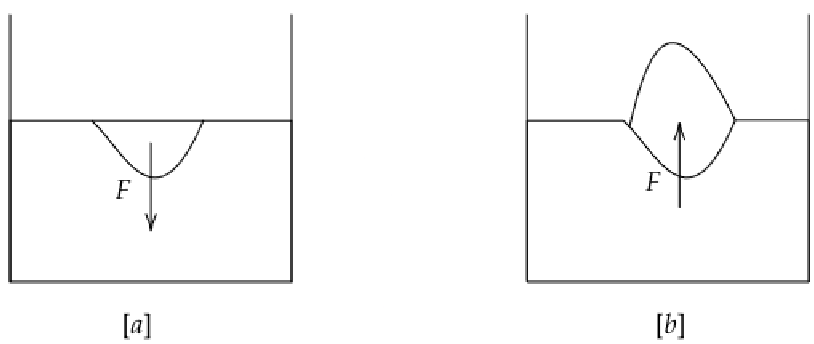

Figure 10: A picture of Archimedes' Law where the weight of a section of fluid has the same magnitude of the buoyant force of that displaced section

There are a few ways to see why this is true. First, there is the intuitive understanding that if this fluid in [a] is in equilibrium, then no part of the fluid is moving. The section we carved out has weight, so the surrounding fluid below it must push back with a force equal to its weight to cancel it out and keep everything in equilibrium. When we replace the fluid section with an object, the fluid below it is still pushing back up with the same force, and that is what becomes the buoyant force. We can also see this more concretely.

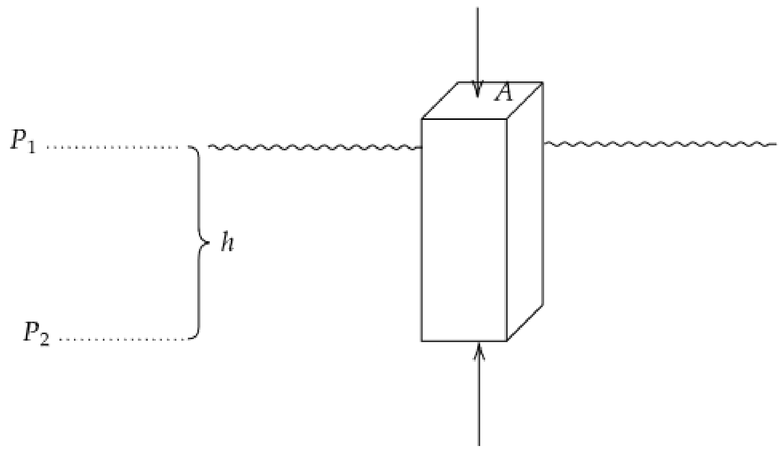

Figure 11: The buoyant force on a rectangular prism

We see in Figure 11 that we a have rectangular prism with area A submerged in water. The distance between the sea level and the bottom of the prism is h. We are going to be working with the assumption that the difference between the barometric pressure at the top of the prism and at sea level is negligible. We have P_1 and P_2 as the two pressures at sea level and at the bottom of the prism, respectively. The last section told us that

$$P_2 = P_1 + \rho g h.$$

There is a force above from the pressure and a force below from the pressure (we know there are actually forces from every direction, but the later forces cancel out, so we ignore them). The buoyant force is simply the difference between the force from the pressure above the prism and the force from the pressure below the prism. It is important to note that the buoyant force only factors in the external forces exerted onto the body from the outside fluid. It doesn't take into account the body's gravitational weight. When we calculate the buoyant force, finding the difference between the forces, we get

$$F_b = \rho g h A.$$

Since $\rho h A$ is $\rho h V$ which is the mass of the displaced fluid, m we see that the buoyant force is simply mg, or the gravitational weight of the displaced fluid. This argument generalizes nicely to irregularly shaped objects. We can simply break up such an object into infinitesimal rectangular prisms and apply the same reasoning as above as pictured in Figure 12.

Hopefully, this argument gives a better understanding and foundation for the basis of Archimedes' Law.

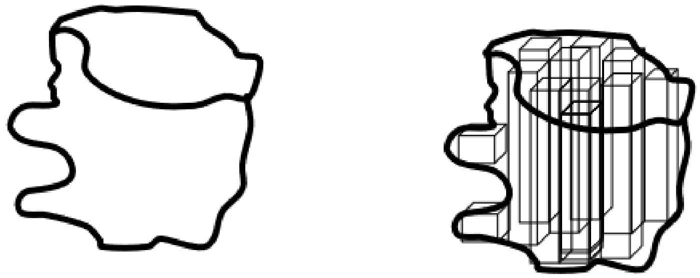

Figure 12: Breaking up an irregular body into infinitesimal rectangular prisms

8 Bernoulli's Equation

Bernoulli's Equation describes in compressible fluids that are flowing at a constant velocity. The typical diagram for this is depicted in Figure 13.

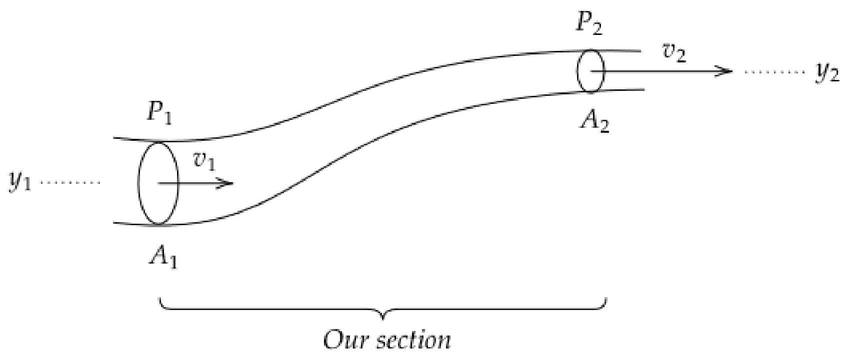

Figure 13: A flow of in compressible fluid.

We have marked off the beginning of a section and the end of that section. At the beginning of our section, the cross sectional area is A_1, the height is y_1[3],

[3]We are taking the height at the center of the cross sectional area. We will be working with the assumption that the difference in the height between particles in the same cross sectional

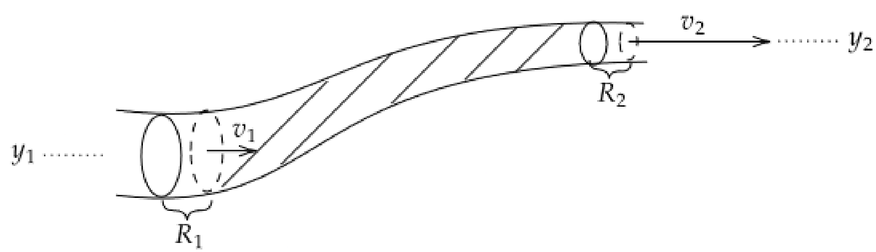

Figure 14: The system after some time, t

the pressure is P_1, and the velocity is A_1. These values are similarly labeled for the end of the section. We are only going to be focusing on our specific section of this flowing fluid. That section is marked off with the brace. It is important to note that there is still fluid behind and in front of our section, and that will become important when we consider external forces and work. One useful thing to keep in mind is that our fluid is in compressible. One consequence of that is that the amount of fluid flowing into a given region must be the amount of fluid exiting that given region. Another way of saying it is that the rate of flow into the region is the same as the rate of flow out of the region. In the case of Figure 13, that means

$$A_1 v_1 = A_2 v_2. \tag{13}$$

This is the equation of continuity. One simple consequence is to see that because A_1 is larger than A_2, V_2 must be larger than v_1. This is means the fluid flows faster in more narrow regions. Now, we are going to consider the energy of our region before and after some time, t. We will also consider the external forces and thus work done on the system during that time, t. We know the the energy before t plus the work done during t must equal the energy after t. After sometime time, t, the fluid, and thus the region, will have traveled some distance. This is pictured in Figure 14.

Our region has moved, and new fluid has then taken its place. We want to consider the difference in the total energy before and after the time t. When considering total energies, we have to sum up the energy of every single particle. This means summing the kinetic energy and potential energy of each individual particle. We'll then take the difference between the sum after t and the sum before t. However, as we see in Figure 14, the shaded region is the same both

area is negligible

before and after t. This isn't to say that it is made up of the same fluid or particles, those particles there before t would have moved, but it means that the particles that replace them after t will have the same velocities and height, and thus the same energy as the particles they replaced. What that means is that because both regions before and after t contain the fluid in the shaded area, and because the fluid in the shaded area both before and after t will have the same energy, we can ignore that in calculation. We can ignore it because it will simply cancel out when we take the difference. Then, all that has changed is that the ends have moved. The region has gained the region R_2 and it has lost the region R_1 in terms of fluid. That means that to find the difference, we must find the energy of R_2 that was gained and subtract it from the energy of R_1 which was lost. Finding this difference, we get that the total change in the energy of the system before and after t is

$$E_f - E_i = m\left(\frac{1}{2}v_2^2 + gy_2\right) - m\left(\frac{1}{2}v_1^2 + gy_1\right) \tag{14}$$

where m is the mass of R_1 and R_2. The region the mass is the same in both is because the mass of R_1 is $\rho A_1 v_1 t$ and the mass of R_2 is $\rho A_2 v_2 t$. These are both equal because of (13). Now, as we've discussed, the change in the energy of the system (the right hand side of (14)) is simply the work done on the region during t. The work being done on the region comes from the fluid behind R_1 and the fluid in front of R_2. The fluid behind R_1 is pushing it, while the fluid in front of R_2 is being pushed by it. Note, we only consider these forces because they are external force since that fluid is outside of the system. The force of the walls do no work since those forces are always orthogonal to the direction of motion. We neglect the forces between the particles because those are internal forces. We can quickly see why we can just neglect those forces.

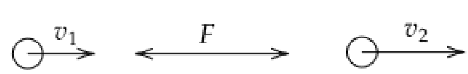

Figure 15: Two particles with a force between them

In Figure 15, we see two particles moving with velocities v_1 and v_2. There is a force between them, F. After some time, t, the force will have done some work. That work will be

$$W = Ft(v_2 - v_1).$$

In our fluid case, the particles are very close by, so for neighboring points, v_2 is approximately equal to v_1 will would mean that the total work done by this internal force is zero. This is why, to a good approximation, we can neglect the contribution of the internal forces to the work. Now, returning back to our system, we say that the net work done is the difference between the work from the back of the region pushing the fluid and the work from the front of the fluid by being pushed by the region. The forces of the pushes are governed by the pressure, and are P_1A_1 and P_2A_2, respectively. The force pushing goes over a distance v_1t while the resisting force of the fluid being pushes goes over a distance v_2t. This means that the total work done on our system is

$$W = P_1A_1v_t - P_2A_2v_2t.$$

We can, once more, utilize (13) to say

$$W = A_1v_1t(P_1 - P_2). \tag{15}$$

We can now equate the right hand sides of (15) and (14), so

$$m\left(\frac{1}{2}v_2^2 + gy_2\right) - m\left(\frac{1}{2}v_1^2 + gy_1\right) = A_1v_1t(P_1 - P_2).$$

We can rewrite this to say

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2.$$

Since our choice of P_1 and P_2 was arbitrary, this must hold for any location of our fluid. We then translate this into the more general case

$$P + \frac{1}{2}\rho v^2 + \rho g y = C \tag{16}$$

where C is a constant. This is Bernoulli's famous equation. One bizarre consequence of it is that it implies P_2 is smaller than P_1 which is not so intuitive given that the particles are flowing faster in that area. Bernoulli's Equation is a form of the conservation of energy, and the term containing pressure represents how pressure actually does work on a fluid. The pressure governs the force a segment of fluid feels from its neighbors, and those forces will change its energy.

References

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